Funnel control of linear systems under output measurement losses

Thomas Berger, Lukas Lanza

***Abstract*—We consider tracking control of linear minimum phase systems with known arbitrary relative degree which are subject to possible output measurement losses. We provide a control law which guarantees the evolution of the tracking error within a (shifted) prescribed performance funnel whenever the output signal is available. The result requires a maximal duration of measurement losses and a minimal time of measurement availability, which both strongly depend on the internal dynamics of the system, and are derived explicitly. The controller is illustrated by a simulation of a mass-on-car system.**

***Index Terms*—linear systems, funnel control, output tracking, measurement losses, minimum phase**

I. Introduction

We study output tracking for linear minimum phase systems with arbitrary relative degree under possible output measurement losses. Such phenomena are of significant practical relevance whenever signals are transmitted over large distances or via digital communication networks and may hence be prone to signal losses or package dropouts. In the presence of output measurement losses the performance of closed-loop control strategies may seriously deteriorate and even lead to instability. In the present paper we present a reliable strategy for linear systems which is able to guarantee a prescribed margin for the tracking error and after any period of possible output measurement losses it is able to recapture the error within this time-varying margin by appropriately shifting it.

Output measurement losses are typically considered within the framework of networked control systems, see e.g. [1]–[4]. Within this approach, event-triggered controllers have been designed in order to guarantee global asymptotic stability, see [5]–[7] for linear systems and [8], [9] for nonlinear systems. H_∞ control approaches have been considered in [10], [11] and model predictive control in [12], [13]. However, as far as the authors are aware, tracking control with prescribed performance bounds for the tracking error has not yet been considered. To achieve this, in the present paper we use the methodology of funnel control.

The concept of funnel control goes back to the seminal work [14], see also the survey in [15]. The funnel controller proved to be the appropriate tool for tracking problems in various applications such as control of industrial servo-systems [16] and underactuated multibody systems [17], [18], control of electrical circuits [19], [20], control of peak inspiratory pressure [21], adaptive cruise control [22], [23] and even the control of infinite-dimensional systems such as a boundary controlled heat equation [24], a moving water tank [25] and defibrillation processes of the human heart [26].

The novel funnel control design that we present in this paper relies on an intrinsic "availability function" which encodes (as a binary value) whether the output measurement is available at some time instant, or if the measurement is lost. As a consequence, no precise *a priori* information about the time instants where the measurement is lost or recaptured is necessary. Then the basic idea for the control design is simply to employ a classical funnel controller on each interval where the output is available, set the input to zero when it is not available and restart the controller when the output signal is received again. Because we restrict ourselves to linear systems no blow-up may occur when the input is zero. The crucial obstacle in the feasibility proof of the control design in our main result Theorem II.1 is to show that the resulting control input in the closed-loop system is globally bounded. To this end, we require appropriate assumptions on the maximal duration of measurement losses and the minimal time of measurement availability, which we summarize in Section I-B. The bounds for these durations essentially depend on the internal dynamics of the system – if the internal dynamics are absent, no restrictions must be made. However, if they are present a key step is to find an invariant set for the internal dynamics and to choose the initial width of the performance funnel large enough – this is elaborated in Section I-C. The control design is illustrated by a simulation of a mass-on-car system in Section III.

A. Nomenclature

Throughout the present article we use the following notation, where $I \subseteq \mathbb{R}$ denotes an interval and $\mathbb{R}_{\geq 0} := [0,\infty)$. $\mathbb{N}$ is the set of positive integers; $\mathbb{C}_- := \{ z \in \mathbb{C} \mid \mathrm{Re} z < 0 \}$; $\|x\| := \sqrt{x^\top x}$ is the Euclidean norm of $x \in \mathbb{R}^n$; $\mathbf{Gl}_n(\mathbb{R})$ is the set of invertible matrices $A \in \mathbb{R}^{n\times n}$; for $A \in \mathbf{Gl}_n(\mathbb{R})$ we write $A > 0$ ($A < 0$) if A is positive (negative) definite; $\sigma(A) \subseteq \mathbb{C}$ is the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$; $\mathcal{L}^\infty(I;\mathbb{R}^p)$ is the Lebesgue space of measurable and essentially bounded functions $f : I \to \mathbb{R}^p$ with norm $\|f\|_\infty := \operatorname{ess\,sup}_{t\in I} \|f(t)\|$; $\mathcal{W}^{k,\infty}(I;\mathbb{R}^p)$ is the Sobolev space of k-times weakly differentiable functions $f : I \to \mathbb{R}^p$ such that $f, \dot f, \ldots, f^{(k)} \in \mathcal{L}^\infty(I;\mathbb{R}^p)$; $\mathcal{C}^k(I;\mathbb{R}^p)$ is the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^p$, $\mathcal{C}(I;\mathbb{R}^p) = \mathcal{C}^0(I;\mathbb{R}^p)$; $f|_J$ is the restriction of $f : I \to \mathbb{R}^n$ to $J \subseteq I$.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 362536361.

Thomas Berger and Lukas Lanza are with the Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany (e-mail: thomas.berger@math.upb.de, lanza@math.upb.de).

B. System class

We consider linear systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B, C^\top \in \mathbb{R}^{n\times m}$; in particular, the dimensions of the input $u(t)$ and the output $y(t)$ coincide. We assume that the system has strict relative degree $r \in \mathbb{N}$, i.e., $CA^kB = 0$ for all $k = 0, \ldots, r-2$, and $\Gamma := CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$. Then, a straightforward generalization of [27, Thm. 3] yields that there exist $R_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, r$, $S, P^\top \in \mathbb{R}^{m\times(n-rm)}$ and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ such that system (1) is equivalent to

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), \\ \dot{\eta}(t) &= Q\eta(t) + Py(t) \end{aligned} \tag{2}$$

with initial conditions

$$\begin{aligned} (y(0), \ldots, y^{(r-1)}(0)) &= (y_0^0, \ldots, y_{r-1}^0) \in \mathbb{R}^{rm}, \\ \eta(0) &= \eta^0 \in \mathbb{R}^{n-rm}. \end{aligned}$$

We introduce the system class under consideration.

Definition I.1. *For $r, m \in \mathbb{N}$ a system* (2) *belongs to the system class $\Sigma_{r,m}$, if*

(i) the high-gain matrix $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$ is sign definite[1]*; w.l.o.g. we assume $\Gamma + \Gamma^\top > 0$,*

(ii) the system is minimum phase, i.e., $\sigma(Q) \subseteq \mathbb{C}_-$.

We write $(A, B, C) \in \Sigma_{r,m}$.

We record the following result, the proof of which is straightforward.

Lemma I.2. *For $L \in \mathbb{R}^{p\times p}$ with $\sigma(L) \subseteq \mathbb{C}_-$ there exists $0 < K = K^\top$ such that $KL + L^\top K = -I_p$, and*

$$\forall\, t \geq 0 : \ \|e^{Lt}\| \leq \sqrt{\|K^{-1}\|\|K\|}\, e^{-\frac{1}{2\|K\|}t}.$$

In virtue of Lemma I.2, for Q from (2) let

$$M := \sqrt{\|K^{-1}\|\|K\|}, \ \mu := \frac{1}{2\|K\|}, \tag{3}$$

where $KQ + Q^\top K = -I_{n-rm}$. If $n - rm = 0$, then we set $M := 0$ and $\mu := 1$.

For later use we record that, for $t \geq t_0 \geq 0$, we have

$$\int_{t_0}^{t} \|e^{Q(s-t_0)}\| \mathrm{d}s \leq \frac{M}{\mu}(1 - e^{-\mu(t-t_0)}) \leq \frac{M}{\mu}, \tag{4a}$$

$$\int_{t_0}^{t} \|e^{Q(s-t_0)}\| \mathrm{d}s \leq M \int_{t_0}^{t} |e^{-\mu(s-t_0)}| \mathrm{d}s \leq M(t - t_0). \tag{4b}$$

Further, we recall that the second of equations (2) has the solution

$$\eta(t) = e^{Q(t-t_0)}\eta(t_0) + \int_{t_0}^{t} e^{Q(t-s)} Py(s)\mathrm{d}s \tag{5}$$

and, moreover, for any signal $y \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ we have

$$\begin{aligned} \|\eta(t)\| \leq{}& Me^{-\mu(t-t_0)}\|\eta(t_0)\| \\ &+ M\|P\|\, \|y|_{[t_0,t]}\|_\infty \int_{t_0}^{t} e^{-\mu(s-t_0)}\mathrm{d}s. \end{aligned} \tag{6}$$

Since we consider situations where the output measurement signal may be lost for some time, we propose assumptions relating the maximal duration of measurement losses and minimal time of measurement availability. The package dropouts in the system and the accompanying lost information of the measurements $y(t)$ are not assumed to happen in *a priori* known time intervals. We only assume that it is possible to determine, at every time instant t, whether the measurement of $y(t)$ is available or not; if the availability is not certain, then it should be rendered "unavailable" (this also encompasses the situation that, after a dropout, the availability of the measurement is only determined with some delay). Based on this we define an "availability function"

$$a(t) = \begin{cases} 1, & \text{measurement of } y(t) \text{ available}, \\ 0, & \text{measurement of } y(t) \text{ not available}. \end{cases} \tag{7}$$

In order to introduce the assumption on the maximal duration of measurement losses and the minimal time of measurement availability we define the sequences (t_k^-), (t_k^+) with $t_k^\pm \nearrow \infty$ and $t_k^- < t_k^+ < t_{k+1}^- < t_{k+1}^+$ such that

$$\begin{aligned} \{\, t \geq 0 \mid a(t) = 1 \,\} &= \bigcup_{k\in\mathbb{N}} (t_k^+, t_{k+1}^-], \\ \{\, t \geq 0 \mid a(t) = 0 \,\} &= \bigcup_{k\in\mathbb{N}} (t_k^-, t_k^+], \end{aligned} \tag{8}$$

this is, on the interval $[t_k^+, t_{k+1}^-)$ the signal is available, and on the interval $[t_k^-, t_k^+)$ the signal is not available. Note, that it is also possible that both sequences contain only finitely many points, then either $a(t) = 1$ for $t \geq t_N^+$ or $a(t) = 0$ for $t \geq t_N^-$ for some $N \in \mathbb{N}$.

Assumption 1. *Let $p := \|P\|$, $s := \|S\|$ and $\beta := 1 + \frac{spM}{\mu} + \sum_{i=1}^{r} \|R_i\|$ be given by the system parameters, M, μ from* (3) *and q, A_r be the constants introduced in Section I-C. The signal is lost for at most $\Delta > 0$, i.e., for $t_k^\pm$ as in* (8) *we have $|t_k^- - t_k^+| \leq \Delta$ for all $k \in \mathbb{N}$, such that for some $\kappa \geq 2$ and $\theta > s$ we have that Δ satisfies*

$$spM\Delta^2 e^{\beta\Delta} \leq 1, \tag{Δ_1}$$

$$pM^2\Delta e^{\beta\Delta} \leq \frac{q}{A_r} \cdot \frac{\mu(\kappa - 1)}{2\kappa\theta}. \tag{Δ_2}$$

Assumption 2. *The signal is available for at least $\delta > 0$, i.e., for $t_k^\pm$ as in* (8) *we have $|t_k^+ - t_{k+1}^-| \geq \delta$ for all $k \in \mathbb{N}$, such that for Δ, β, κ, θ from Assumption 1 and M, μ from* (3) *we have that δ satisfies*

$$e^{\mu\delta} \geq 2\kappa M\big(M + p\Delta e^{\beta\Delta}(1 + sM^2\Delta)\big), \tag{δ_1}$$

$$e^{\mu\delta} \geq 2\frac{\kappa}{\theta}(1 + sM^2). \tag{δ_2}$$

Remark I.3. *For systems with trivial internal dynamics (the second equation in* (2) *is not present) Assumptions 1 & 2 are much weaker. More precisely, in this case we have $p = 0$,*

[1] That is, for any $v \in \mathbb{R}^m$ we have $v^\top \Gamma v = 0$ if, and only if, $v = 0$.

$s = 0$ and $M = 0$ with which the inequalities (Δ_1), (Δ_2) and (δ_1), (δ_2) are always satisfied (for $\theta = 2\kappa$) and hence arbitrary $\Delta > 0$ and $\delta > 0$ are possible so that $|t_k^- - t_k^+| \le \Delta$ and $|t_k^+ - t_{k+1}^-| \ge \delta$ for all $k \in \mathbb{N}$. So the only (implicit) requirement is that the sequence $(|t_k^- - t_k^+|)$ is bounded.

C. Control objective, design parameters and feedback law

1) Control objective: We aim to find a control scheme which achieves tracking of a given reference trajectory with prescribed transient behavior of the error, where the measurement output is subject to dropouts. To be more precise, for a system (2) with $(A, B, C) \in \Sigma_{r,m}$ and a given reference signal $y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ the output y tracks the reference in the sense that, whenever the measurement of y is available to the controller, the error $e := y - y_{\text{ref}}$ evolves within a prescribed *performance funnel*

$$\mathcal{F}_\varphi := \{ (t, e) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \},$$

where φ belongs to the following set of monotonically increasing functions

$$\Phi := \left\{ \phi \in \mathcal{C}^1(\mathbb{R}_{\ge 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \forall\, t_2 \ge t_1 \ge 0 :\ 0 < \phi(t_1) \le \phi(t_2), \\ \exists\, d > 0\ \forall\, t \ge 0 : \\ |\dot{\phi}(t)| \le d(1 + \phi(t)) \end{array} \right\}.$$

The performance funnel $\mathcal{F}_\varphi$ joins the two objectives of $e(t)$ approaching zero with prescribed transient behavior and asymptotic accuracy. Its boundary is given by the reciprocal of φ, see also Fig. 2. We stress that φ may be unbounded and in this case (and if no measurement losses occur for $t \ge T$ for some $T > 0$) asymptotic tracking may be achieved, i.e., $\lim_{t\to\infty} e(t) = 0$.

2) Design parameters: In order to formulate the control law, which achieves the control objective, we introduce the following design parameters. In Fig. 1 the five steps towards the choice of the design parameters $\eta^* \in \mathbb{R}$ and $\varphi_0 \in \Phi$ are depicted.

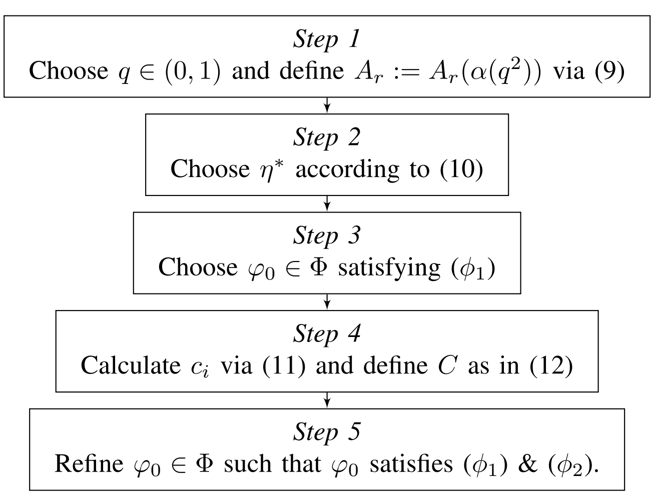


Fig. 1: Flowchart for the choice of the controller design parameters.

Step 1. Choose $q \in (0, 1)$ and define the bijection $\alpha : [0, 1) \to [1, \infty)$ via $\alpha(s) = 1/(1 - s)$. For $k \ge 0$ define the function

$$A_k(s) = \sum\nolimits_{j=0}^{k} s^j, \tag{9}$$

and set $A_r := A_r(\alpha(q^2))$.

Step 2. For $\Delta, \delta, p, s, \beta, \kappa, \theta$ from Assumptions 1 & 2, respectively, $x_{\text{ref}}(\cdot) := (y_{\text{ref}}(\cdot), \dot{y}_{\text{ref}}(\cdot), \ldots, y_{\text{ref}}^{(r-1)}(\cdot))$, and M, μ from (3) choose $\eta^* > 0$ with

$$\eta^* \ge \max \left\{ \begin{array}{c} \frac{p}{\mu}\|y_{\text{ref}}\|_\infty e^{\mu\delta}, \|x_{\text{ref}}\|_\infty e^{\mu\delta}, \\ \frac{\|x_{\text{ref}}\|_\infty (1+e^{\beta\Delta}) e^{\mu\delta - \beta\Delta}}{\Delta} \end{array} \right\}, \tag{10}$$

and set $E := \theta \Delta e^{\beta\Delta} \eta^* > 0$.

Step 3. Let $\varphi_0 \in \Phi$ such that

$$\varphi_{0,\min} := \frac{2\kappa p M^2}{\mu(\kappa - 1)\eta^*} \le \varphi_0(0) \le \frac{q}{A_r E} =: \varphi_{0,\max}, \tag{ϕ_1}$$

which is possible by (Δ_2).

Step 4. Now, we choose some additional constants which are necessary to exploit [15, Cor. 1.10]. Let $\hat{\alpha}^\dagger(z) = z/(1 + z)$ which obviously yields $\hat{\alpha}^\dagger(s\alpha(s)) = s$, and define $\tilde{\alpha}(s) = 2s\alpha'(s) + \alpha(s) = (1+s)/(1-s)^2$. Further, let $\mu_0 := \frac{d(1+\varphi_0(0))}{\varphi_0(0)}$ where $d > 0$ is due to properties of Φ and observe that $\operatorname{ess\,sup}_{t\ge 0}(|\dot{\varphi}_0(t)|/\varphi_0(t)) \le \mu_0$; here we use this possibly larger constant μ_0 to guarantee that it only depends on $\varphi_0(0)$. Then, in virtue of [15, Eq. (12)], for $k = 1, \ldots, r - 1$ we recursively define the constants $c_0 = 0$ and

$$\begin{aligned} e_1^0 &:= \varphi_0(0) e(0), \\ c_1 &:= \max\{\|e_1^0\|^2, \hat{\alpha}^\dagger(1 + \mu_0), q^2\}^{1/2} < 1, \\ \mu_k &:= 1 + \mu_0\big(1 + c_{k-1}\alpha(c_{k-1}^2)\big) \\ &\qquad + \tilde{\alpha}(c_{k-1}^2)\big(\mu_{k-1} + c_{k-1}\alpha(c_{k-1}^2)\big), \\ e_k^0 &:= \varphi_0(0) e^{(k-1)}(0) + \alpha(\|e_{k-1}^0\|^2) e_{k-1}^0, \\ c_k &:= \max\{\|e_k^0\|^2, \hat{\alpha}^\dagger(\mu_k), q^2\}^{1/2} < 1, \end{aligned} \tag{11}$$

where $e^{(i)}(0) = y_i^0 - y_{\text{ref}}^{(i)}(0)$ for $i = 0, \ldots, r - 1$, and set

$$C := \sum_{i=1}^{r-1} c_i + c_{i-1}\alpha(c_{i-1}^2) + (1 + c_{r-1}\alpha(c_{r-1}^2)). \tag{12}$$

Step 5. We refine the function $\varphi_0 \in \Phi$ satisfying (ϕ_1) such that for an intermediate $\rho \in (0, \delta)$

$$\varphi_0(\rho) \ge \max \left\{ \frac{C e^{\mu\delta}}{\eta^*}, \frac{C e^{\mu\delta}}{\Delta \eta^*} \right\}. \tag{ϕ_2}$$

Remark I.4. *We note that the purpose of the constant q chosen in Step 1 of the design procedure is to determine the initial width of the performance funnel, described by the upper bound for $\varphi_0(0)$ in (ϕ_1). Then again, condition (ϕ_2) ensures that its width (and hence the tracking error) is not too large before the signal possibly vanishes the next time.*

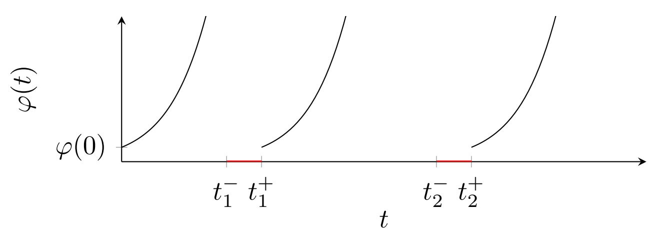
3) Feedback law: The idea for the controller design is to choose a funnel function $\varphi_0 \in \Phi$ (as in the previous subsection) which is reset whenever $a(t) = 0$. Then, as soon as $a(t^*) = 1$ for some $t^* \ge 0$ and the measurement is available again, the funnel controller from [15] is restarted with $\varphi(t) = \varphi_0(t - t^*)$ so that $\varphi(t^*) > 0$ and the performance funnel is sufficiently large at t^* to ensure applicability of [15, Thm. 1.9]. For feasibility we assume that the availability function $a(\cdot)$ from (7) is left-continuous and has only finitely many jumps in each compact interval. With this, and recalling $\alpha(s) = 1/(1-s)$, we introduce the following control law for systems (2) under possible output measurement losses:

$$\begin{aligned} \tau(t) &= \begin{cases} t, & a(t) = 0, \\ \tau(t-), & a(t) = 1, \end{cases} \\ \varphi(t) &= \begin{cases} 0, & a(t) = 0, \\ \varphi_0(t - \tau(t)), & a(t) = 1, \end{cases} \\ e_1(t) &= \varphi(t)e(t) = \varphi(t)\big(y(t) - y_{\mathrm{ref}}(t)\big), \\ e_{i+1}(t) &= \varphi(t)e^{(i)}(t) + \alpha(\|e_i(t)\|^2)e_i(t),\ \ i = 1, \ldots, r-1, \\ u(t) &= -a(t)\alpha(\|e_r(t)\|^2)e_r(t). \end{aligned} \tag{13}$$

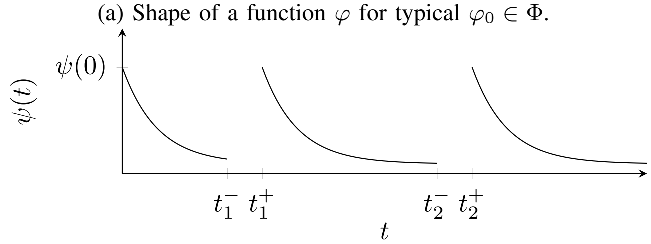
Note that if $\Gamma + \Gamma^\top < 0$ the control would read $u(t) = a(t)\alpha(\|e_r(t)\|^2)e_r(t)$.

If the output measurement is always available, i.e., $a(t) = 1$ for all $t \geq 0$, then the controller (13) coincides with that proposed in [15] and the existence of a global solution of the closed-loop system follows from the results presented there. Since it is not known *a priori* when output measurement losses occur, the funnel function φ cannot be globally defined in advance. Therefore, φ is defined online as part of the control law (13); it is equal to a shifted version of the reference funnel function φ_0 whenever measurements are available, and zero otherwise. Note that the loss of the system's output signal possibly introduces a discontinuity in the control signal.

A typical choice for a funnel function is $\varphi_0(t) = (ae^{-bt} + c)^{-1}$ with $a, b, c > 0$, which is depicted in Fig. 2.



(a) Shape of a function φ for typical $\varphi_0 \in \Phi$.



(b) Corresponding funnel boundary $\psi(t) = 1/\varphi(t)$.

Fig. 2: Schematic shape of a typical funnel boundary with shifts.

II. Main result

In the following main result we show that the application of the funnel controller (13) to a system (2) under possible output measurement losses leads to a closed-loop initial-value problem which has a global solution. By a solution of (2), (13) on $[0, \omega)$ we mean a function $(y, \eta) \in \mathcal{C}^{r-1}([0, \omega), \mathbb{R}^m) \times \mathcal{C}([0, \omega), \mathbb{R}^{n-rm})$ with $\omega \in (0, \infty]$, which satisfies $(y(0), \ldots, y^{(r-1)}(0)) = (y_0^0, \ldots, y_{r-1}^0)$, $\eta(0) = \eta^0$ and $(y^{(r-1)}, \eta)|_{[0,\omega)}$ is locally absolutely continuous and satisfies the differential equation in (2) with u defined by (13) for almost all $t \in [0, \omega)$; (y, η) is called maximal, if it has no right extension that is also a solution.

Theorem II.1. *Consider a system* (2) *with* $(A, B, C) \in \Sigma_{r,m}$ *and initial values* $(y_0^0, \ldots, y_{r-1}^0) \in \mathbb{R}^{rm}$ *and* $\eta^0 \in \mathbb{R}^{n-rm}$. *Let* $y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, $a(\cdot)$ *be an availability function as in* (7) *which is left-continuous and has only finitely many jumps in each compact interval, and choose design parameters* η^* *as in* (10), *and* $\varphi_0 \in \Phi$ *satisfying* (ϕ_1),(ϕ_2). *If the initial conditions*

$$\forall\, i = 1, \ldots, r : \ \|e_i(0)\| < 1, \tag{14a}$$

$$\|\eta^0\| \leq \eta^* \tag{14b}$$

are satisfied, then the control scheme (13) *applied to system* (2) *yields an initial value problem which has a solution, every solution can be extended to a maximal solution and every maximal solution* $(y, \eta) : [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-rm}$ *has the following properties:*

(i) the solution is global, i.e., $\omega = \infty$,

(ii) the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ *evolves within the funnel boundaries, i.e.,* $\varphi(t)\|e(t)\| < 1$ *for all* $t \geq 0$,

(iii) the control signal is globally bounded, i.e., $u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, *and* $y \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

The proof is relegated to Appendix B. Note that the proof is constructive and we provide an explicit global bound for the control input u.

III. Simulation

To illustrate the action of the proposed controller, we numerically simulate an application of the funnel control scheme (13) to a system (2). We consider the *mass-on-car* system introduced in [28], where on a car with mass m_1 (in kg) a ramp is mounted on which a mass m_2 (in kg), coupled to the car by a spring-damper-component with spring constant $k > 0$ (in N/m) and damping $d > 0$ (in Ns/m), passively moves; a control force $F = u$ (in N) can be applied to the car. The situation is depicted in Fig. 3. The equations of motion for

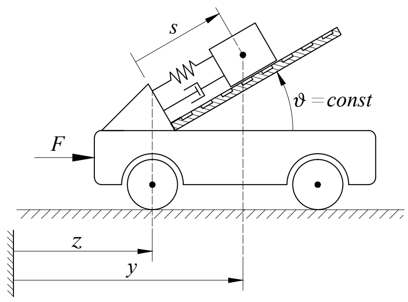


Fig. 3: Mass-on-car system.

the system read

$$\begin{bmatrix} m_1+m_2 & m_2\cos(\vartheta) \\ m_2\cos(\vartheta) & m_2 \end{bmatrix} \begin{pmatrix} \ddot z(t) \\ \ddot s(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t)+d\dot s(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{15a}$$

with the horizontal position of the second mass m_2 as output

$$y(t) = z(t) + \cos(\vartheta)s(t). \tag{15b}$$

For the simulation we choose the parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $d = 1$, $\vartheta = \pi/4$ and the initial values $z(0) = s(0) = \dot z(0) = \dot s(0) = 0$. As a reference signal we choose $y_{\rm ref} : \mathbb{R}_{\ge 0} \to \mathbb{R}$, $t \mapsto \cos(t)$, by which $\|y_{\rm ref}\|_\infty = \|x_{\rm ref}\|_\infty = 1$. As elaborated in [15, Sec. 3], for the above parameters system (15) has relative degree two with respect to the output (15b), and hence belongs to $\Sigma_{2,1}$. Thus, it can equivalently be written in the form (2) with $r = 2$ and

$$R_1 = 0,\ R_2 = \tfrac{8}{9},\ S = \tfrac{-4\sqrt{2}}{9}\begin{bmatrix} 2 & 1 \end{bmatrix},\ \Gamma = \tfrac{1}{9},$$
$$Q = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix},\ P = 2\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

According to Assumptions 1 & 2 with $q = 0.95$, $\kappa = 15$, $\theta = (1 + 0.01)s$, $\mu = 0.3305$, $M = 2.2477$ we assume $\Delta \le 2.4 \cdot 10^{-3}$ s and $\delta \ge 15.6$ s; and (10) is satisfied with $\eta^* = 141\,764$. We choose $\varphi_0(t) = (ae^{-bt} + c)^{-1}$. According to ($\phi_1$) the funnel function has to satisfy

$$\varphi_{0,\min} = 6.5360 \cdot 10^{-4} \le \varphi_0(0) \le 6.5360 \cdot 10^{-4} = \varphi_{0,\max},$$

and we choose $c = 0.08$, $a = 1/\varphi_{0,\min} - c$ and $b = 1$. Then, the constant from (12) is given as $C = 21.4683$, and condition (ϕ_2) is satisfied with $\varphi(\rho) = 12$, where $\rho = 0.99\,\delta$.

We simulate output tracking over the interval $0-45$ seconds. The simulation has been performed in MATLAB (solver: ode23tb). For illustration purposes we consider two losses and reappearances of the output signal. Fig. 4 shows the error $e = y - y_{\rm ref}$ between the system's output and the reference signal. As expectable the error evolves within the prescribed funnel boundaries whenever the output signal is available, and remains bounded whenever the signal is not available. In Fig. 5 the control input is depicted. It can be seen that on large time intervals, especially after t_1^- and t_2^-, the input signal is zero. Only when the performance funnel gets tighter again a large control action is necessary, which induces some small peaks in the input when a small tracking error is enforced. But even in the presence of measurement losses the control input is bounded and the evolution of the tracking error within the (shifted) performance funnel is guaranteed.

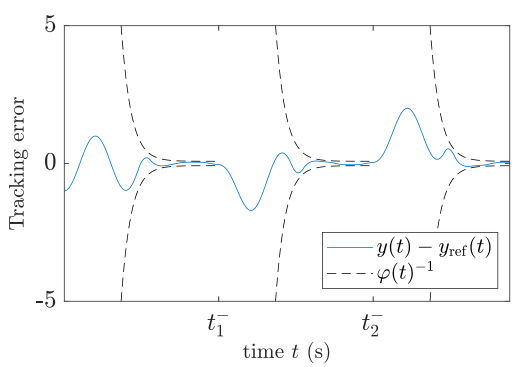


Fig. 4: Error between the output y and the reference signal $y_{\rm ref}$; and funnel boundary $1/\varphi$.

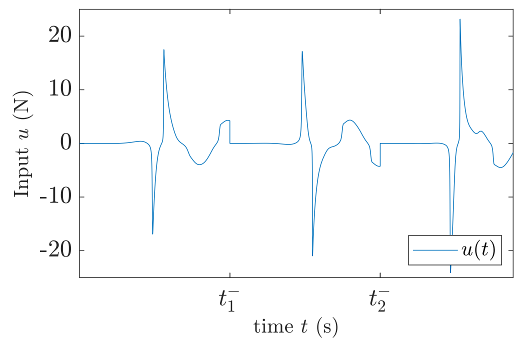


Fig. 5: Control input u.

IV. Conclusion

In the present paper we introduced a novel funnel controller for output reference tracking of linear minimum phase systems which are prone to losses of the output measurements. We proved that the closed-loop system has a global solution, and the presented feedback law achieves a prescribed transient behaviour of the tracking error within a (shifted) performance funnel and all involved signals are bounded; in particular, the input signal is bounded. Feasibility of the control requires a maximal duration of measurement losses Δ and a minimal time of measurement availability δ, for both of which upper and lower bounds, resp., have been derived explicitly. However, these bounds are conservative (as can be seen by the numerical example in Section III) and further research is necessary to find better estimates.

Another topic for future research is the extension of the results to nonlinear systems. Regarding this, it is clear that some kind of Lipschitz condition is required for the system, because otherwise a blow-up of the solutions cannot be excluded on time-intervals where the output measurement is not available.

Appendix

A. Technical lemmas

We derive a lemma which provides an exponential bound for the solution whenever no measurement is available.

Lemma A.1. *Consider a linear system* (2) *with* $(A, B, C) \in \Sigma_{r,m}$. *Then for* M, μ *from* (3), β *as in Assumption 1 and* $s = \|S\|$ *we have that for all solutions* $(y, \eta) \in \mathcal{C}^{r-1}([0,\omega), \mathbb{R}^m) \times \mathcal{C}([0,\omega), \mathbb{R}^{n-rm})$, $\omega \in (0, \infty]$, *of* (2) *with* $u|_{(t_0,t_1)} = 0$ *for* $0 \le t_0 < t_1 \le \omega$ *and with* $x = (y^\top, \dot y^\top, \dots, (y^{(r-1)})^\top)^\top$ *that for all* $t \in [t_0, t_1)$

$$\|x|_{[t_0,t]}\|_\infty \le \left(\|x(t_0)\| + sM\|\eta(t_0)\| \int_{t_0}^t e^{-\mu(\tau - t_0)} \mathrm{d}\tau \right) e^{\beta(t-t_0)}.$$

Proof. Let $x = (x_1^\top, \dots, x_r^\top)^\top$ and set $w(t) := \|x|_{[t_0,t]}\|_\infty$ for $t \in [t_0, \omega)$. Then we have that

$$\dot x(t) = \begin{pmatrix} x_2(t) \\ \vdots \\ x_r(t) \\ \sum_{i=1}^r R_i x_i(t) + S\eta(t) \end{pmatrix}$$

for almost all $t \in [t_0, t_1]$ and upon integration we obtain

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^{t} \|x(\tau)\| + \sum_{i=1}^{r} \|R_i\| \|x_i(\tau)\| + s\|\eta(\tau)\| \,\mathrm{d}\tau.$$

Then, using (4a) and (5), we have

$$\begin{aligned} w(t) &\le \|x(t_0)\| + \sup_{r\in[t_0,t]} \int_{t_0}^{r} \Bigg[w(\tau) + s\|e^{Q(\tau-t_0)}\eta(t_0)\| \\ &+ \sum_{i=2}^{r} \|R_i\| \|x_i|_{[t_0,\tau]}\|_\infty + \left(\|R_1\| + \frac{spM}{\mu} \right) \|x_1|_{[t_0,\tau]}\|_\infty \Bigg] \mathrm{d}\tau \\ &\le \|x(t_0)\| + \int_{t_0}^{t} \underbrace{\left[1 + \left(\sum_{i=1}^{r} \|R_i\| + \frac{spM}{\mu} \right) \right]}_{=\beta} w(\tau) \,\mathrm{d}\tau \\ &\quad + sM\|\eta(t_0)\| \int_{t_0}^{t} e^{-\mu(\tau-t_0)} \,\mathrm{d}\tau. \end{aligned}$$

The assertion then follows from Grönwall's lemma. □

The second lemma provides a technical estimate for the proof of the main result.

Lemma A.2. *For $k = 0, \ldots, r$, $r \in \mathbb{N}$, let A_k be given as in (9). Let $\alpha : [0, 1) \to [1, \infty)$ be a bijection, $q \in (0, 1)$ and $\lambda, E \ge 0$ with*

$$\lambda \le \frac{q}{A_r(\alpha(q^2))E}. \tag{16}$$

Further let $\xi_0, \ldots, \xi_{r-1} \in \mathbb{R}^n$ with

$$\forall\, k \in \{0, \ldots, r-1\} : \ \|\xi_k\| \le E. \tag{17}$$

Then define $\zeta_0 := 0$ and $\zeta_{k+1} \in \mathbb{R}^n$ for $k = 0, \ldots, r-1$ by

$$\zeta_{k+1} := \lambda\xi_k + \alpha(\|\zeta_k\|^2)\zeta_k. \tag{18}$$

Then

$$\forall\, k \in \{1, \ldots, r\} : \ \|\zeta_k\| \le \lambda E A_{k-1}(\alpha(q^2)) \le q.$$

Proof. First observe that for $s \ge 0$ we have

$$\forall\, k \in \mathbb{N} : A_k(s) \le A_k(s) + s^{k+1} = A_{k+1}(s).$$

Furthermore, for $\tilde{A}_k := A_k\big(\alpha(q^2)\big)$ we have that

$$\lambda E \tilde{A}_k \le \lambda E A_r(\alpha(q^2)) \overset{(16)}{\le} q.$$

Finally, we show that

$$\forall\, k \in \{1, \ldots, r\} : \ \|\zeta_k\| \le \lambda E \tilde{A}_{k-1} \tag{19}$$

by induction over k. For $k = 1$ we have

$$\|\zeta_1\| \overset{(18)}{\le} \lambda\|\xi_0\| \overset{(17)}{\le} \lambda E.$$

Let (19) be true for some $k \in \{1, \ldots, r-1\}$. Then, we obtain, using monotonicity of $\alpha(\cdot)$

$$\begin{aligned} \|\zeta_{k+1}\| &\overset{(18)}{\le} \lambda\|\xi_k\| + \alpha(\|\zeta_k\|^2)\|\zeta_k\| \\ &\overset{(17),(19)}{\le} \lambda E + \alpha\big((\lambda E \tilde{A}_{k-1})^2\big)\lambda E \tilde{A}_{k-1} \\ &\le \lambda E\big(1 + \alpha(q^2)\tilde{A}_{k-1}\big) \\ &= \lambda E\big(1 + \alpha(q^2)A_{k-1}(\alpha(q^2))\big) = \lambda E A_k(\alpha(q^2)), \end{aligned}$$

where we have used that $1 + sA_{k-1}(s) = A_k(s)$. This proves (19). □

B. Proof of Theorem II.1

Proof. Step 1. First, we establish the existence of a solution of (2), (13). With x_{ref} as defined in Section I-C and following Step 1 in the proof of [15, Thm. 1.9], we introduce $\mathcal{B} = \{\, w \in \mathbb{R}^m \mid \|w\| < 1 \,\}$ and for $\alpha(s) = 1/(1-s)$ the map

$$\gamma : \mathcal{B} \to \mathbb{R}^m, \ w \mapsto \alpha(\|w\|^2)w,$$

and with this the sets $\mathcal{D}_k$ and maps $\rho_k : \mathcal{D}_k \to \mathcal{B}$, $k = 1, \ldots, r$ recursively as follows:

$$\begin{aligned} &\mathcal{D}_1 := \mathcal{B}, \ \rho_1 : \mathcal{D}_1 \to \mathcal{B}, \ \zeta_1 \mapsto \zeta_1, \\ &\mathcal{D}_k := \left\{ (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^{km} \,\middle|\, \begin{array}{l} Z := (\zeta_1, \ldots, \zeta_{k-1}) \in \mathcal{D}_{k-1}, \\ \zeta_k + \gamma(\rho_{k-1}(Z)) \in \mathcal{B} \end{array} \right\}, \\ &\rho_k : \mathcal{D}_k \to \mathcal{B}, \ (\zeta_1, \ldots, \zeta_k) \mapsto \zeta_k + \gamma(\rho_{k-1}(\zeta_1, \ldots, \zeta_{k-1})). \end{aligned}$$

With this we define the set

$$\mathcal{D} := \{\, (t, \xi) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^{rm} \mid \varphi(t)\|\xi - x_{\mathrm{ref}}(t)\| \in \mathcal{D}_r \,\}$$

and $\rho : \mathcal{D} \to \mathcal{B}$, $(t, \xi) \mapsto \rho_r\big(\varphi(t)\big(\xi - x_{\mathrm{ref}}(t)\big)\big)$. Since $a(\cdot)$ is left-continuous the set $\mathcal{D}$ is relatively open. Then, u in (13) satisfies

$$u(t) = -a(t)\alpha(\|\rho(t, x(t))\|^2)\rho(t, x(t))$$

and we formally define the function $F : \mathcal{D} \times \mathbb{R}^{n-rm} \to \mathbb{R}^n$ by

$$\begin{aligned} F(t, \xi_1, \ldots, \xi_r, \eta) = \Big(&\xi_2, \ldots, \xi_r, \\ &\sum_{i=1}^{r} R_i\xi_i + S\eta - a(t)\alpha(\|\rho(t,\xi)\|^2)\rho(t,\xi), Q\eta + P\xi_1\Big) \end{aligned}$$

and obtain with $x(\cdot) := (y(\cdot), \dot{y}(\cdot), \ldots, y^{(r-1)}(\cdot))$ an initial value problem

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\eta}(t) \end{pmatrix} &= F\left(t, x(t), \eta(t)\right), \\ x(0) &= \big(y_0^0, \ldots, y_{r-1}^0\big), \ \eta(0) = \eta^0, \end{aligned} \tag{20}$$

which is equivalent to (2),(13). Note that F is continuous in $(\xi_1, \ldots, \xi_r, \eta)$ and locally essentially bounded and, in particular, measurable in the variable t regardless of the possible discontinuities of $a(\cdot)$. Therefore, since $(0, x(0)) \in \mathcal{D}$, a straightforward adaption of [29, Thm. B.1] to the current context yields the existence of a maximal solution $(x, \eta) : [0, \omega) \to \mathbb{R}^n$ of (20), where $\omega \in (0, \infty]$. Moreover, the closure of the graph of the solution of (20) is not a compact subset of $\mathcal{D} \times \mathbb{R}^{n-rm}$.

Step 2. Next, we establish (ii) on $[0, \omega)$. To this end, let (t_k^-), (t_k^+) be as in (8). It is also possible that both sequences contain only finitely many points, then either $a(t) = 1$ for $t \ge t_N^+$ or $a(t) = 0$ for $t \ge t_N^-$ for some $N \in \mathbb{N}$; the following arguments apply, *mutatis mutandis*, in both cases. We define $\mathbf{e}(\cdot) := x(\cdot) - x_{\mathrm{ref}}(\cdot)$. Since we consider a subclass of the system class under consideration in [15], and since by (14a) we have $\varphi(0)\mathbf{e}(0) \in \mathcal{D}_r$, the result [15, Thm. 1.9] restricted to the interval $[0, t_1^-]$ is applicable and ensures assertion (ii) for $t \in [0, t_1^-] \subseteq [0, \omega)$, the inclusion since without measurement losses [15, Thm. 1.9] yields $\omega = \infty$. Further, since by construction we have $\varphi|_{[t_1^-, t_1^+)} = 0$ assertion (ii) is true for

$t \in [t_1^-, t_1^+) \subseteq [0,\omega)$, the inclusion via standard theory of (linear) differential equations since $u|_{[t_1^-, t_1^+)} \equiv 0$. In order to reapply [15, Thm. 1.9] at $t = t_1^+$, we establish that the initial conditions (14) are satisfied for $t = t_1^+$. First, we show (14a) at t_1^+. We set $\psi(\cdot) := 1/\varphi_0(\cdot)$. By (5) and (6), the initial condition (14b) and using (10) we have

$$\begin{aligned} \|\eta(t_1^-)\| &\overset{(4a)}{\le} M e^{-\mu\delta}\eta^* + \frac{pM}{\mu}\left(\psi(0) + \|y_{\mathrm{ref}}\|_\infty\right) \\ &\overset{(10)}{\le} 2Me^{-\mu\delta}\eta^* + \frac{pM}{\mu}\psi(0). \end{aligned} \tag{21}$$

By [15, Cor. 1.10] we have for all $i = 0,\dots,r-2$ and the constants defined in (11) that

$$\forall\, t \in [0, t_1^-) \;:\; \|e^{(i)}(t)\| \le \psi(t)(c_{i+1} + c_i\alpha(c_i^2)), \tag{22}$$

and moreover, since $\|e_r(t)\| \le 1$ for $t \in [0, t_1^-)$, we have $\|e^{(r-1)}(t)\| \le \psi(t)(1 + c_{r-1}\alpha(c_{r-1}^2))$. Hence, $\mathbf{e}(t) \le C\psi(t)$ for C defined in (12), and in particular

$$\mathbf{e}(t_1^-) \le C\psi(t_1^-) \le C\psi(\rho) \tag{23}$$

for $\rho < \delta \le t_1^-$ as in (ϕ_2) since ψ is monotonically decreasing by properties of Φ. With this, using Lemma A.1 we obtain

$$\begin{aligned} \|x|_{[t_1^-, t_1^+]}\|_\infty &\le \Bigg(\|x(t_1^-)\| \\ &\quad + sM\|\eta(t_1^-)\| \int_{t_1^-}^{t_1^+} e^{-\mu(s - t_1^-)} \mathrm{d}s \Bigg) e^{\beta(t_1^+ - t_1^-)} \\ &\overset{(4b)}{\le} \Bigg(\|\mathbf{e}(t_1^-)\| + \|x_{\mathrm{ref}}\|_\infty + sM\Delta\|\eta(t_1^-)\| \Bigg) e^{\beta(t_1^+ - t_1^-)} \\ &\overset{(23)}{\le} \|x_{\mathrm{ref}}\|_\infty e^{\beta\Delta} + \left(C\psi(\rho) + sM\Delta\|\eta(t_1^-)\|\right) e^{\beta\Delta} \end{aligned} \tag{24}$$

and therefore

$$\begin{aligned} \|\mathbf{e}(t_1^+)\| &\le \|x_{\mathrm{ref}}(t_1^+)\| + \|x(t_1^+)\| \le \|x_{\mathrm{ref}}\|_\infty + \|x|_{[t_1^-, t_1^+]}\|_\infty \\ &\overset{(24)}{\le} \|x_{\mathrm{ref}}\|_\infty\left(1 + e^{\beta\Delta}\right) + \left(C\psi(\rho) + sM\Delta\|\eta(t_1^-)\|\right) e^{\beta\Delta} \\ &\overset{(10),(\phi_2)}{\le} \Delta e^{\beta\Delta}\eta^* e^{-\mu\delta} + \Delta e^{\beta\Delta}\eta^* e^{-\mu\delta} + sM\Delta e^{\beta\Delta}\|\eta(t_1^-)\| \\ &\overset{(21)}{=} 2\Delta\eta^* e^{\beta\Delta - \mu\delta}\left(1 + sM^2\right) + \Delta e^{\beta\Delta}\frac{spM^2}{\mu}\psi(0) \\ &\overset{(\phi_1)}{\le} 2\Delta\eta^* e^{\beta\Delta - \mu\delta}\left(1 + sM^2\right) + s\Delta e^{\beta\Delta}\frac{\kappa - 1}{\kappa}\eta^* \\ &\overset{(\delta_2)}{\le} \Delta e^{\beta\Delta}\frac{\theta}{\kappa}\eta^* + s\Delta e^{\beta\Delta}\frac{\kappa - 1}{\kappa}\eta^* \\ &\overset{s \le \theta}{\le} \theta\Delta e^{\beta\Delta}\eta^* = E, \end{aligned} \tag{25}$$

and hence in particular,

$$\|e^{(i)}(t_1^+)\| < E, \quad i = 0,\dots,r-1.$$

Therefore, invoking (ϕ_1), Lemma A.2 (applied with $\lambda = \varphi(t_1^+) = \varphi_0(0)$) yields

$$\begin{aligned} &\|e_i(t_1^+)\| \le q \le c_i < 1, \; i = 1,\dots,r-1, \\ &\|e_r(t_1^+)\| \le q, \end{aligned} \tag{26}$$

hence $\varphi(t_1^+)\mathbf{e}(t_1^+) \in \mathcal{D}_r$. Furthermore, via (21), using Lemma A.1 and (23) we obtain with similar estimates as above

$$\begin{aligned} \|\eta(t_1^+)\| &\overset{(6),(4b)}{\le} M\|\eta(t_1^-)\| + pM\|y|_{[t_1^-, t_1^+]}\|_\infty\Delta \\ &\overset{(24)}{\le} M\|\eta(t_1^-)\| + pM\Delta\left(\|x_{\mathrm{ref}}\|_\infty e^{\beta\Delta} \right.\\ &\qquad \left. + \left(C\psi(\rho) + sM\Delta\|\eta(t_1^-)\|\right)e^{\beta\Delta}\right) \\ &\overset{(21)}{\le} 2M^2 e^{-\mu\delta}\eta^* + \frac{pM^2}{\mu}\psi(0) \\ &\quad + pM\Delta\left(C\psi(\rho) + \|x_{\mathrm{ref}}\|_\infty\right)e^{\beta\Delta} \\ &\quad + spM^2\Delta^2\left(2Me^{-\mu\delta}\eta^* + \frac{pM}{\mu}\psi(0)\right)e^{\beta\Delta} \\ &\overset{(10),(\phi_2)}{\le} 2M^2e^{-\mu\delta}\eta^*(1 + spM\Delta^2e^{\beta\Delta}) + 2pM\Delta e^{\beta\Delta - \mu\delta}\eta^* \\ &\quad + \frac{pM^2}{\mu}\left(1 + spM\Delta^2 e^{\beta\Delta}\right)\psi(0) \\ &\overset{(\Delta_1),(\phi_1)}{\le} \frac{\kappa - 1}{\kappa}\eta^* + 2e^{-\mu\delta}\eta^*\left(M^2 + pM\Delta e^{\beta\Delta}(1 + sM^2\Delta)\right) \\ &\overset{(\delta_1)}{\le} \frac{\eta^*}{\kappa} + \frac{\kappa - 1}{\kappa}\eta^* = \eta^*. \end{aligned} \tag{27}$$

Therefore, the initial conditions (14) are satisfied at $t = t_1^+$ and [15, Thm. 1.9] is applicable for $t \ge t_1^+$. Moreover, invoking (26) the estimates (21), (25) and (27) are valid for $t = t_2^-$ and $t = t_2^+$, respectively, since $\|\eta(t_1^+)\| \le \eta^*$ and $[t_1^+, t_2^+] \subseteq [0,\omega)$ via the same arguments as above. So we obtain inductively

$$\begin{aligned} &\varphi(t_k^+)\mathbf{e}(t_k^+) \in \mathcal{D}_r \text{ and } \|\eta(t_k^+)\| \le \eta^* \text{ so (14) is satisfied} \\ &\Rightarrow \text{funnel control applicable for } t \in [t_k^+, t_{k+1}^-) \subseteq [0,\omega) \\ &\overset{(10),(\phi_1),(\phi_2)}{\Rightarrow} \|\eta(t_{k+1}^-)\| \text{ satisfies (21)} \\ &\overset{(25)}{\Rightarrow} \|\mathbf{e}(t_{k+1}^+)\| \le E \\ &\qquad \overset{(26)}{\Rightarrow} \|e_i(t_{k+1}^+)\| \le q < 1, \; i = 1,\dots,r \\ &\qquad \overset{(\phi_1)}{\Rightarrow} \varphi(t_{k+1}^+)\mathbf{e}(t_{k+1}^+) \in \mathcal{D}_r \\ &\overset{(27)}{\Rightarrow} \|\eta(t_{k+1}^+)\| \le \eta^*. \end{aligned}$$

This means, the funnel control can be reapplied at $t = t_k^+$ for all $k \in \mathbb{N}$ with $[t_k^+, t_{k+1}^-) \subseteq [0,\omega)$. This yields (ii) on $[0,\omega)$.

Step 3. We show $y \in \mathcal{W}^{r,\infty}([0,\omega);\mathbb{R}^m)$ and $u \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$. Invoking (22) and (24) we obtain $y \in \mathcal{W}^{r-1,\infty}([0,\omega);\mathbb{R}^m)$. To obtain a global bound for u and $y^{(r)}$ let $Y_{\max} = \max_{i=0,\dots,r}\|y_{\mathrm{ref}}^{(i)}\|_\infty$, $\lambda := \inf_{t\ge 0}\psi(t)$, $\gamma > 0$ such that $\frac{1}{2}v^\top(\Gamma + \Gamma^\top)v \ge \gamma\|v\|^2$ for all $v \in \mathbb{R}^m$, and recall $\tilde{\alpha}(s) = (1+s)/(1-s)^2$. Further set

$$\bar{\eta} := \max\left\{\eta^*, M\eta^* + \frac{pM}{\mu}\left(\psi(0) + Y_{\max}\right)\right\}$$

and observe that $\|\eta(t)\| \le \bar{\eta}$ for all $t \in [t_k^+, t_{k+1}^-]$ by a similar estimate as in (21) and that $\|\eta(t)\| \le \eta^* \le \bar{\eta}$ by a similar estimate as in (27) for $t \in [t_k^-, t_k^+]$. Define with c_i from (11)

$$\begin{aligned} \tilde{C} &:= \mu_0\left(1 + \tfrac{c_{r-1}}{1 - c_{r-1}^2}\right) + \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \tfrac{c_{r-1}}{1 - c_{r-1}^2}\right) \\ &\quad + \sum_{i=1}^r \|R_i\|\left(1 + \tfrac{c_{i-1}}{1 - c_{i-1}^2} + \tfrac{Y_{\max}}{\lambda}\right) + \tfrac{s}{\lambda}\bar{\eta} + \tfrac{Y_{\max}}{\lambda}. \end{aligned}$$

Let $\varepsilon \in (0,1)$ be the unique point such that $\frac{\tilde{C}}{\gamma\varphi_0(0)} = \frac{\varepsilon}{1-\varepsilon^2}$. Then, we define

$$c_r := \max\left\{\|e_r^0\|^2, \varepsilon, q^2\right\}^{1/2} < 1.$$

We show that $\|e_r(t)\| \le c_r$ for all $t \in [0, t_1^-)$. Suppose there exists $t_1 \in [0, t_1^-)$ such that $\|e_r(t_1)\| > c_r$ and define

$$t_0 := \max\left\{ t \in [0, t_1] \mid \|e_r(t)\| = c_r \right\},$$

which is well-defined since $\|e_r(0)\| \le c_r$. First observe that, by the same calculations as in the proof of [15, Cor. 1.10], we have for $\gamma_{r-1}(t) := \alpha(\|e_{r-1}(t)\|^2)e_{r-1}(t)$ that

$$\|\dot{\gamma}_{r-1}(t)\| \le \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \alpha(c_{r-1}^2)c_{r-1}\right).$$

Furthermore, since $\|e_r(t)\| \ge c_r$ for all $t \in [t_0, t_1]$ we have $\alpha(\|e_r(t)\|^2) \ge 1/(1-c_r^2)$ and hence

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_r(t)\|^2 = e_r(t)^\top\left(\dot{\varphi}(t)e^{(r-1)}(t) + \varphi(t)e^{(r)}(t) + \dot{\gamma}_{r-1}(t)\right)\\
&\le \|e_r\|\left(\mu_0\varphi(t)\|e^{(r-1)}(t)\| + \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\right.\\
&\quad\left. + \varphi(t)Y_{\max} + \varphi(t)\left(\sum_{i=1}^r \|R_i\|\|y^{(i-1)}(t)\| + s\bar{\eta}\right)\right)\\
&\quad - \tfrac{1}{2}\varphi(t)\alpha(\|e_r(t)\|^2)e_r(t)^\top(\Gamma+\Gamma^\top)e_r(t)\\
&\le \|e_r\|\left(\mu_0\varphi(t)\|e^{(r-1)}(t)\| + \tilde{\alpha}(c_{r-1}^2)\left(\mu_{r-1} + \frac{c_{r-1}}{1-c_{r-1}^2}\right)\right.\\
&\quad\left. + \frac{Y_{\max}}{\lambda} + \sum_{i=1}^r \|R_i\|\left(1 + \frac{c_{i-1}}{1-c_{i-1}^2} + \frac{Y_{\max}}{\lambda}\right) + \frac{s}{\lambda}\bar{\eta}\right)\\
&\quad - \frac{\gamma\varphi(0)}{1-c_r^2}\|e_r(t)\|^2\\
&\le \left(\tilde{C} - \gamma\varphi(0)\frac{c_r}{1-c_r^2}\right)\|e_r(t)\| \le 0,
\end{aligned}$$

by which $c_r < \|e_r(t_1)\| \le \|e_r(t_0)\| = c_r$, a contradiction. By (26) we have that $\|e_r(t_k^+)\| \le q \le c_r$ for all $k \in \mathbb{N}$ with $t_k^+ \in [0,\omega)$. Therefore, the arguments above can be reapplied on any interval $[t_k^+, t_{k+1}^-) \subseteq [0,\omega)$ to achive $\|e_r(t)\| \le c_r$ for all $t \in [t_k^+, t_{k+1}^-)$. Then, invoking $u|_{[t_k^-, t_k^+)} = 0$, it follows from (13) that $\|u(t)\| \le c_r/(1-c_r^2)$ for all $t \in [0,\omega)$, thus $u \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$. As a consequence, it follows from (2) that $y^{(r)} \in \mathcal{L}^\infty([0,\omega);\mathbb{R}^m)$.

Step 4. Next we show that the solution is global. Suppose the opposite, i.e., $\omega < \infty$. Then, since $\|\eta(t)\| \le \bar{\eta}$ and for all $i = 1,\dots,r$ we have $\|e_i(t)\| \le c_i < 1$ for $t \in [t_k^+, t_{k+1}^-)$ by [15, Cor. 1.10] and Step 3, and $\|e_i(t)\| \le q \le c_i$ for $t \in [t_k^-, t_k^+)$ by (26) (note that it is straightforward to extend the estimate (25) to $t \in [t_k^-, t_k^+)$), it follows that the closure of the graph of the solution of (20) is a compact subset of $\mathcal{D} \times \mathbb{R}^{n-rm}$, which contradicts the findings of Step 1. This yields assertion (i) and consequently assertions (ii) & (iii) follow. This completes the proof. □

References

[1] M. García-Rivera and A. Barreiro, "Analysis of networked control systems with drops and variable delays," *Automatica*, vol. 43, no. 12, pp. 2054–2059, 2007.

[2] Y.-L. Wang and G.-H. Yang, "Time delay and packet dropout compensation for networked control systems: a linear estimation method," *Int. J. Control*, vol. 83, no. 1, pp. 115–124, 2010.

[3] M. Cloosterman, L. Hetel, N. van de Wouw, W. Heemels, J. Daafouz, and H. Nijmeijer, "Controller synthesis for networked control systems," *Automatica*, vol. 46, no. 10, pp. 1584–1594, 2010.

[4] D. Nešić and A. Teel, "Input–output stability properties of networked control systems," *IEEE Trans. Autom. Control*, vol. 49, no. 10, pp. 1650–1667, 2004.

[5] D. Lehmann and J. Lunze, "Event-based control with communication delays and packet losses," *Int. J. Control*, vol. 85, no. 5, pp. 563–577, 2012.

[6] R. Blind and F. Allgöwer, "On the stabilizability of continuous-time systems over a packet based communication system with loss and delay," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 6466–6471, 2014.

[7] S. Linsenmayer, D. V. Dimarogonas, and F. Allgöwer, "Periodic event-triggered control for networked control systems based on non-monotonic Lyapunov functions," *Automatica*, vol. 106, pp. 35–46, 2019.

[8] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 586–601, 2011.

[9] V. Dolk and M. Heemels, "Event-triggered control systems under packet losses," *Automatica*, vol. 80, pp. 143–155, 2017.

[10] H. Gao and T. Chen, "Network-based $\mathcal{H}_\infty$ output tracking control," *IEEE Trans. Autom. Control*, vol. 53, no. 3, pp. 655–667, 2008.

[11] Y. Tang, H. Gao, and J. Kurths, "Robust $\mathcal{H}_\infty$ self-triggered control of networked systems under packet dropouts," *IEEE Trans. Cybernetics*, vol. 46, no. 12, pp. 3294–3305.

[12] D. M. de la Pena and P. D. Christofides, "Lyapunov-based model predictive control of nonlinear systems subject to data losses," *IEEE Trans. Autom. Control*, vol. 53, no. 39, pp. 2076–2089, 2008.

[13] M. Lješnjanin, D. E. Quevedo, and D. Nešić, "Packetized MPC with dynamic scheduling constraints and bounded packet dropouts," *Automatica*, vol. 50, no. 3, pp. 784–797, 2014.

[14] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, "Tracking with prescribed transient behaviour," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 7, pp. 471–493, 2002.

[15] T. Berger, A. Ilchmann, and E. P. Ryan, "Funnel control of nonlinear systems," *Mathematics of Control, Signals, and Systems*, vol. 33, no. 1, pp. 151–194, feb 2021.

[16] C. M. Hackl, *Non-identifier Based Adaptive Control in Mechatronics–Theory and Application*, ser. Lecture Notes in Control and Information Sciences. Cham, Switzerland: Springer-Verlag, 2017, vol. 466.

[17] T. Berger, S. Drücker, L. Lanza, T. Reis, and R. Seifried, "Tracking control for underactuated non-minimum phase multibody systems," *Nonlinear Dynamics*, vol. 104, no. 4, pp. 3671–3699, 2021.

[18] T. Berger, S. Otto, T. Reis, and R. Seifried, "Combined open-loop and funnel control for underactuated multibody systems," *Nonlinear Dynamics*, vol. 95, pp. 1977–1998, 2019.

[19] T. Berger and T. Reis, "Zero dynamics and funnel control for linear electrical circuits," *J. Franklin Inst.*, vol. 351, no. 11, pp. 5099–5132, 2014.

[20] A. Senfelds and A. Paugurs, "Electrical drive DC link power flow control with adaptive approach," in *Proc. 55th Int. Sci. Conf. Power Electr. Engg. Riga Techn. Univ., Riga, Latvia*, 2014, pp. 30–33.

[21] A. Pomprapa, S. Weyer, S. Leonhardt, M. Walter, and B. Misgeld, "Periodic funnel-based control for peak inspiratory pressure," in *Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan*, 2015, pp. 5617–5622.

[22] T. Berger and A.-L. Rauert, "A universal model-free and safe adaptive cruise control mechanism," in *Proceedings of the MTNS 2018*, Hong Kong, 2018, pp. 925–932.

[23] ——, "Funnel cruise control," *Automatica*, vol. 119, Article 109061, 2020.

[24] T. Reis and T. Selig, "Funnel control for the boundary controlled heat equation," *SIAM J. Control Optim.*, vol. 53, no. 1, pp. 547–574, 2015.

[25] T. Berger, M. Puche, and F. L. Schwenninger, "Funnel control for a moving water tank," *Automatica*, vol. 135, Article 109999, 2022.

[26] T. Berger, T. Breiten, M. Puche, and T. Reis, "Funnel control for the monodomain equations with the FitzHugh-Nagumo model," *J. Diff. Eqns.*, vol. 286, pp. 164–214, 2021.

[27] A. Ilchmann and F. Wirth, "On minimum phase," *Automatisierungstechnik*, vol. 12, pp. 805–817, 2013.

[28] R. Seifried and W. Blajer, "Analysis of servo-constraint problems for underactuated multibody systems," *Mech. Sci.*, vol. 4, pp. 113–129, 2013.

[29] A. Ilchmann and E. P. Ryan, "Performance funnels and tracking control," *Int. J. Control*, vol. 82, no. 10, pp. 1828–1840, 2009.