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# SRB MEASURES FOR ANOSOV ACTIONS

YANNICK GUEDES BONTHONNEAU, COLIN GUILLARMOU, AND TOBIAS WEICH

ABSTRACT. Given a general Anosov $\mathbb{R}^\kappa$ action on a closed manifold, we study properties of certain invariant measures that have recently been introduced in [GBGHW20] using the theory of Ruelle-Taylor resonances. We show that these measures share many properties of Sinai-Ruelle-Bowen measures for general Anosov flows such as smooth disintegrations along the unstable foliation, positive Lebesgue measure basins of attraction and a Bowen formula in terms of periodic orbits. Finally we show that if the action in the positive Weyl chamber is transitive, the measure is unique and has full support.

## INTRODUCTION

On a closed, smooth Riemannian manifold $(\mathcal{M}, g)$ (normalized with volume 1) we consider a locally free abelian action $\tau : \mathbb{R}^\kappa \to \mathrm{Diffeo}(\mathcal{M})$. Assume that $\tau$ is Anosov, and denote by $\mathcal{W} \subset \mathbb{R}^\kappa$ the maximal cone of tranversally hyperbolic elements (see Section 1.1 for a precise definition of all these terms). In [GBGHW20] it was proved that there exists a Radon probability measure $\mu$, called *the physical measure*, such that for every function $f \in C^0(\mathcal{M})$ and every open proper[1] subcone $\mathcal{C} \subset \mathcal{W}$,

$$\mu(f) = \lim_{T\to+\infty} \frac{1}{|\mathcal{C}_T|} \int_{A\in\mathcal{C}_T} \int_{\mathcal{M}} f(\tau(-A)(x))dxdA. \tag{0.1}$$

Here $\mathcal{C}_T = \{A \in \mathcal{C} \mid e(A) \leq T\}$, for some linear form $e$ on $\mathbb{R}^\kappa$, positive on $\mathcal{W}$. In this article we will explore the properties of the measure $\mu$, proving in particular:

**Theorem 1.** *Let $\tau$ be a transitive, smooth, locally free, $\mathbb{R}^\kappa$ Anosov action. Let $\mu$ be an invariant Radon probability measure on $\mathcal{M}$, then the following conditions are equivalent:*

*(1) $\mu$ is the physical measure.*
*(2) For every continuous $f$, every open proper subcone $\mathcal{C} \subset\subset \mathcal{W}$, and Lebesgue almost every $x \in \mathcal{M}$,*

$$\mu(f) = \lim_{T\to+\infty} \frac{1}{|\mathcal{C}_T|} \int_{A\in\mathcal{C}_T} \int_{\mathcal{M}} f(\tau(-A)(x))dA$$

*(3) $\mu$ has an absolute continuous disintegration w.r.t. to the local stable foliation, $W^s_{\mathrm{loc}}$.*
*(4) the measure $\mu$ has wave-front set $\mathrm{WF}(\mu) \subset E_s^*$.*

*Such a measure $\mu$ is always ergodic. If in addition we assume that the action is positively transitive in the sense of Definition 2.9, then $\mathrm{supp}(\mu) = \mathcal{M}$.*

These properties are very similar to the properties of the SRB measure for transitive Anosov flows, which was studied extensively by Sinai, Bowen and Ruelle [Sin68, Bow74, Rue76, BR75]. As for all our results, we also obtain a more general and more detailed version (Theorem 3) without the transitivity assumption.

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[1] proper meaning that $\partial\mathcal{C} \cap \partial\mathcal{W} = \{0\}$

For a given smooth Anosov flow, the structural stability implies that any small perturbation of the flow is again an Anosov flow. Furthermore for any fixed Anosov flow, one can associate with each potential $V$ a so-called invariant Gibbs measure that has positive entropy. The world of smooth Anosov flows is thus very rich and, for a fixed flow, there is also a rich ergodic theory due to the different Gibbs measures. In contrast, for higher rank Anosov action the sitation is conjectured (and partially known) to be very rigid: in [KS94] Katok and Spatzier proved that for a list of algebraic Anosov actions, called *standard Anosov actions*, any small perturbation of the Anosov action is Hölder conjugate to the original action. More generally, Katok and Spatzier conjectured (see [Has07, Conjecture 16.8]) that whenever a higher rank Ansosov action cannot be factored into a product of an Anosov flow with another action, they are algebraic in the sense that they come from quotients of symmetric spaces or Lie groups. Despite some important recent advances (see e.g. Spatzier-Vinhage [SV19]) this conjecture is still widely open.

Assuming that this rigidity conjecture holds, the classification of invariant measures reduces to analyzing homogeneous dynamics, i.e $\mathbb{R}^\kappa$ invariant measures on homogeneous spaces. Such measure classifiactions in homogeneous dynamics have been intensively studied in the past decades, starting with the works of Katok and Spatzier [KS95, KS98] and culminating in more recent works of Einsiedler, Katok and Lindenstrauss [EKL06, EL15].

However, in order to make progress in the direction of Katok-Spatzier rigidity conjecture, it is obviously important to understand as many dynamical properties of Anosov actions as possible, without assuming that these actions are homogeneous. In particular it is important to understand and to construct meaningful invariant measures[2]. Let us mention some related results in this direction: in [KKRH11], Kalinin-Katok-Rodriguez Hertz obtain the following: for a locally free abelian Anosov action with $\dim \mathcal{M} = 2\kappa+1$ with $\kappa \geq 2$, an invariant ergodic measure $\mu$ which has positive entropy for some $A \in \mathbb{R}^\kappa$ is absolutely continuous under certain assumptions on the Lyapunov exponents and hyperplanes of $\mu$ (it is thus the same as our SRB measure). Let us finally mention that independently, Carrasco-Rodriguez Hertz [CRH] have constucted an SRB measure using the thermodynamic formalism and also proved the absolute continuity of the conditional measures. For general Anosov actions by smooth Lie groups they show that this is the equilibrium measure associated to the potential given by the unstable Jacobian, as in the rank 1 case.

The second main result of our article concerns the distribution of regular periodic orbits for higher rank actions. We obtain a Bowen-like [Bow72, PP90] formula for the measure $\mu$. A point $x \in \mathcal{M}$ is said to be a periodic point if there exists $A \in \mathbb{R}^\kappa$ such that $\tau(A)(x) = x$. Periodic orbits may have a complicated shape in general, but it is well known that if $\tau(A_0)(x) = x$ for some $A_0 \in \mathcal{W}$, then the orbit set $T = T_x := \{\tau(A)(x) \in \mathcal{M} \,|\, A \in \mathbb{R}^\kappa\}$ is a $\kappa$-dimensional torus – we say that the orbit is regular. We denote by $\mathcal{T}$ the set of such periodic tori of $\tau$ and, for $T \in \mathcal{T}$, we denote by $L(T) := \{A \in \mathbb{R}^\kappa \,|\, \tau(A)(x) = x\}$ the associated lattice.

**Theorem 2.** *Let $\tau$ be a transitive $\mathbb{R}^\kappa$-Anosov action, with Weyl chamber $\mathcal{W}$. Let $\mathcal{C} \subset \mathcal{W}$ be a proper subcone and $\eta \in \mathbb{R}^{\kappa *}$ a dual element that is positive on a slightly larger conic neighbourhood of $\mathcal{C}$. Define $\mathcal{C}_{a,b} := \{A \in \mathcal{C} \,|\, \eta(A) \in [a,b]\}$ if $a, b > 0$. Let $\mu$ be the SRB*

[2]As explained to us by Ralf Spatzier, the existence of ergodic measures with full support is an important tool in the direction of proving the rigidity conjecture (see e.g. [KS07] where this assumption is crucially used, as well as the discussions in [SV19])

*measure and $a, b > 0$. Then for each $f \in C^\infty(\mathcal{M})$, we have*

$$\mu(f) = \lim_{N\to\infty} \frac{1}{|\mathcal{C}_{aN,bN}|} \sum_{T\in\mathcal{T}} \sum_{A\in\mathcal{C}_{aN,bN}\cap L(T)} \frac{\int_T f}{|\det(1-\mathcal{P}_A)|} \tag{0.2}$$

*where $\mathcal{P}_A$ is the linearized Poincaré map of the periodic orbit $A$ restricted to $E_u \oplus E_s$.*

This result is proved using microlocal methods inspired by Dyatlov-Zworski [DZ16] in the rank 1 case: one needs in our setting to combine the Guillemin trace formula with the analysis of the wave-front set of a certain meromorphic function $F_\lambda(X_1, \dots, X_\kappa)$ of the family of commuting vector fields $(X_1, \dots, X_\kappa)$ generating the Anosov action, and this function has a simple pole at $\lambda = 0$ with residue given by $\mu(f)$. The result (0.2) shows some equidistribution of the periodic orbits just as in the rank 1 case, except that here the periodic orbits come as $\kappa$-dimensional tori. Notice that for the case of the Weyl chamber flow on a locally symmetric space $\mathcal{M} = \Gamma\backslash G/M$, the SRB measure is the Haar measure (by uniqueness), thus Theorem 2 gives an expression of the Haar measure in terms of periodic tori: by (4.2), there is $\epsilon > 0$ so that for all $A \in L(T) \cap \mathcal{C}$, $\det(1-\mathcal{P}_A) = e^{2\rho(A)}(1 + \mathcal{O}(e^{-\epsilon|A|}))$ where $\rho$ is the half sum of the positive roots, therefore (0.2) reduces to

$$\mu(f) = \lim_{N\to\infty} \frac{1}{|\mathcal{C}_{aN,bN}|} \sum_{T\in\mathcal{T}} \sum_{A\in\mathcal{C}_{aN,bN}\cap L(T)} e^{-2\rho(A)} \int_T f. \tag{0.3}$$

We notice that even for locally symmetric space where $\mu$ is Haar measure, the formula (0.2) and (0.3) were not proved, and our result is new even in that setting.

As a rather direct consequence of (0.2) we get the following result on the counting of periodic tori:

**Corollary 0.1.** *Assume there is a linear form $\eta \in \mathbb{R}^{\kappa *}$ that is positive on $\mathcal{W}$ and such that for any proper subcone $\mathcal{C} \subset \mathcal{W}$ there is $\varepsilon > 0$ such that $|\det(1-\mathcal{P}_A)| = e^{\eta(A)}(1 - \mathcal{O}(e^{-\varepsilon|A|}))|$ for all $A \in \mathcal{C}$. For any proper subcone $\mathcal{C} \subset \mathcal{W}$ let $\mathcal{C}_N := \{A \in \mathcal{C}, \eta(A)/\|\eta\| \leq N\}$ then*

$$\lim_{N\to\infty} \frac{1}{N} \log\Big( \sum_{T\in\mathcal{T}} \sum_{A\in L(T)\cap\mathcal{C}_N} \mathrm{vol}(T)\Big) = \|\eta\|.$$

Note that the assumptions are fulfilled for all standard Anosov actions (as introduced in [KS94, Sec. 2]). In the special case of Weyl chamber flows, Spatzier [Spa83] proved a related result when the cone $\mathcal{C}$ is the whole Weyl chamber: more precisely he proved that if $s(T) := \min\{|A| \,|\, A \in L(T) \cap \mathcal{W}\}$ denotes the regular systole of a periodic torus $T$, then

$$\lim_{N\to\infty} \frac{1}{N} \log\Big( \sum_{T\in\mathcal{T}, s(T)\leq N} \mathrm{Vol}(T)\Big) = 2\|\rho\|.$$

Recall from above that for Weyl chamber flows one has $\eta = 2\rho$ and $2\|\rho\|$ corresponds also to the topological entropy of the associated geodesic flow. The same asymptotics for torus orbits of Weyl chamber flows has been obtained by Deitmar [Dei04] (yet with slightly different counting region) using trace formulae on higher rank locally symmetric spaces and Lefschetz formulae.

As a byproduct of the proof of Theorem 2, we also construct some zeta-like functions (see Theorem 5). For each function $\psi \in C_c^\infty(\mathcal{W})$ with small enough support, we obtain a function $d_\psi(\lambda)$ holomorphic on $\mathbb{C}^\kappa$ that vanishes exactly when $\hat{\psi}(\lambda - \zeta) = 1$ for some Taylor-Ruelle resonance $\zeta$ of the action (as was introduced in [GBGHW20]). Here $\hat{\psi}$ is the Laplace transform of $\psi$. As far as we know this is the first example of a globally holomorphic zeta-like function for higher rank actions.

Let us mention two results that are related to the Bowen formula in Theorem 2 for the special case of the Anosov action being a Weyl chamber flow on a compact locally symmetric space: Knieper [Kni05] studies the measure of maximal entropy for geodesic flows on compact locally symmetric spaces and showed its uniqueness. From this uniqueness he derives a Bowen formula for $\epsilon$-separated geodesics. Furthermore, Einsiedler, Lindenstrauss, Michel and Venkatesh studied distribution of torus orbits of Weyl chamber flows in [ELMV09, ELMV11]. In the special case of Weyl chamber flows on $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ they obtain a strong equidistribution result of periodic torus orbits [ELMV11, Theorem 1.4] that among others would imply the Bowen type formula above[3]. In [ELMV09] the authors also study torus orbits on certain compact locally symmetric spaces that are constructed from orders in central simple algebras. They also obtain equidistribution results (see [ELMV09, Corollary 1.7]) which are, however, weaker then those obtained for $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ and they seem not to imply Theorem 2 for this special class of compact locally symmetric spaces.

Before closing this introduction, let us briefly mention the tools and techniques we employ in this work. We build on our previous work [GBGHW20] using microlocal methods in the spirit of Faure-Sjöstrand and Dyatlov-Zworski [FS11, DZ16] in the framework of anisotropic spaces (developped originally in dynamical systems by Blank, Keller, Liverani, Baladi, Tsujii, Gouëzel, Butterley [BKL02, GL06, BL07, BT07]). These techniques have a successful history in the context of Anosov flows, and we use them intensively in this work. For the proof of Theorem 1, it is sufficient to be familiar with the notion of Hörmander wavefront set. For the proof of Theorems 2 and 5, however, we assume that the reader is somehow familiar with the work [DZ16] or [FRS08]. We will also be using some classical techniques from the study of dynamical foliations (absolute continuity, Rokhlin disintegrations...).

**Outline of the paper.**

In section 1.1 we give the definition of $\mathbb{R}^\kappa$-Anosov actions and introduce some related basic notations.

In Section 1.2 we collect and discuss crucial properties of the stable and unstable foliations related to Anosov actions which we shall need in the sequel. In particular we give a proof that the conditional densities of Lebesgue measure along the weak-(un)stable foliations are smooth along the orbits. While this fact seems folklore, we couldn't find a precise reference and as we crucially need this in order to apply our microlocal methods, we took the effort to work this out in details.

In Section 1.3 we recall how invariant measures for Anosov actions can be constructed using the spectral theory of Ruelle-Taylor resonances as presented in [GBGHW20]. We also prove some new statements in this context such as Proposition 1.14 that will allow us to show that the measures defined by spectral theory are always absolutely continuous along the stable foliation.

Section 2 and Section 3 are the core of the paper: Section 2 contains the proofs for the different equivalent characterisations of SRB measures (Theorem 1 respectively the more general version Theorem 3) whereas in Section 3 we prove the Bowen formula (Theorem 2).

[3] Note however that our result does not hold for $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ due to the non compactness of this space

Finally in Section 4 we shortly discuss the applications to counting of periodic tori.

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## 1. Anosov actions and dynamical foliations

1.1. **Anosov actions.** Let $(\mathcal{M}, g)$ be a closed, smooth Riemannian manifold and denote by $v_g$ its Riemannian measure which we assume to be normalized with volume 1. Note that while $g$ is fixed, all the results we shall discuss will be independent of the choice of $g$. Let $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ be a locally free action an abelian Lie group $\mathbb{A} \cong \mathbb{R}^\kappa$ on $\mathcal{M}$. Let $\mathfrak{a} := \mathrm{Lie}(\mathbb{A}) \cong \mathbb{R}^\kappa$ be the associated commutative Lie algebra and $\exp : \mathfrak{a} \to \mathbb{A}$ the Lie group exponential map. After identifying $\mathbb{A} \cong \mathfrak{a} \cong \mathbb{R}^\kappa$, this exponential is the identity, but it will be useful to have a notation that distinguishes between transformations $\mathbb{A}$ and infinitesimal transformations $\mathfrak{a}$. Taking the derivative of the $\mathbb{A}$-action one obtains an injective Lie algebra homomorphism

$$X : \begin{cases} \mathfrak{a} & \to C^\infty(\mathcal{M}; T\mathcal{M}) \\ A & \mapsto X_A := \frac{d}{dt}_{|t=0} \tau(\exp(At)) \end{cases} \tag{1.1}$$

which we call the infinitesimal action. By commutativity of $\mathfrak{a}$, $\mathrm{ran}(X) \subset C^\infty(\mathcal{M}; T\mathcal{M})$ is a $\kappa$-dimensional subspace of commuting vector fields. Since the action is locally free, they span a $\kappa$-dimensional smooth subbundle which we call the *neutral subbundle* $E^0 \subset T\mathcal{M}$. It is tangent to the $\mathbb{A}$-orbits on $\mathcal{M}$. We will often study the one-parameter flow generated by a vector field $X_A$ which we denote by $\varphi_t^A$. One has the obvious identity $\varphi_t^A = \tau(\exp(At))$ for $t \in \mathbb{R}$. The Riemannian metric on $\mathcal{M}$ induces norms on $T\mathcal{M}$ and $T^*\mathcal{M}$, both denoted by $\|\cdot\|$.

**Definition 1.1.** An element $A \in \mathfrak{a}$ and its corresponding vector field $X_A$ are called *transversely hyperbolic* if there is a continuous splitting

$$T\mathcal{M} = E_0 \oplus E_u \oplus E_s, \tag{1.2}$$

that is invariant under the flow $\varphi_t^A$ and such that there are $\nu > 0, C > 0$ with

$$\|d\varphi_t^A v\| \le C e^{-\nu|t|} \|v\|, \quad \forall v \in E_s,\ \forall t \ge 0, \tag{1.3}$$

$$\|d\varphi_t^A v\| \le C e^{-\nu|t|} \|v\|, \quad \forall v \in E_u,\ \forall t \le 0. \tag{1.4}$$

We say that the $\mathbb{A}$-action is *Anosov* if there exists an $A_0 \in \mathfrak{a}$ such that $X_{A_0}$ is transversely hyperbolic.

We define the dual bundles $E_u^*, E_s^*, E_0^* \subset T^*\mathcal{M}$ by[4]

$$E_u^*(E_u \oplus E_0) = 0, \quad E_s^*(E_s \oplus E_0) = 0, \quad E_0(E_u \oplus E_s) = 0. \tag{1.5}$$

Given a transversely hyperbolic element $A_0 \in \mathfrak{a}$ we define the *positive Weyl chamber* $\mathcal{W} \subset \mathfrak{a}$ to be the set of $A \in \mathfrak{a}$ which are transversely hyperbolic with the same stable/unstable bundle as

[4]Note that $E_{s/u}^*$ are not the usual dual bundles of $E_{s/u}$ that vanish on $E_{u/s} \oplus E_0$. The notation that we use has grown historically in the semiclassical approach to Ruelle resonances and is justified by the fact that the symplectic lift of $\tau$ to $T^*\mathcal{M}$ is expanding in the $E_u^*$ fibre and contracting in the $E_s^*$ fibre.

$A_0$. The following statement is well known – a proof can for example be found in [GBGHW20, Lemma 2.2].

**Lemma 1.2.** *Given an Anosov action and a transversely hyperbolic element $A_0 \in \mathfrak{a}$, the positive Weyl chamber $\mathcal{W} \subset \mathfrak{a}$ is an open convex cone.*

Note that there are different concrete constructions of Anosov actions and we refer to [KS94, Section 2.2] for examples.

1.2. **Dynamical foliations and absolute continuity.** Since the point of this article is to study in detail the SRB measure of Anosov actions, we will have to consider disintegration of measures along stable and unstable foliations. For this kind of consideration, it will be crucial that these foliations are *absolutely continuous*. This fact is well established. However, for our purposes, we will need that some conditional densities are $C^\infty$. This seems to be folklore, but we have not found a complete proof written down. We have thus decided to recall the relevant definitions, and explain how the regularity of the conditional measures can be derived from existing results in the literature.

**Definition 1.3.** Let $F$ be a partition of $\mathcal{M}$ and given $m \in \mathcal{M}$ let $F(m)$ be the unique element in $F$ containing $m$. Given a neighbourhood $U$ of $m$ denote by $F_{\mathrm{loc}}(m)$ the connected component of $F(m) \cap U$ containing $m$.

The partition $F$ is called a *continuous (resp. Hölder) foliation with $n$-dimensional $C^k$-leaves* if for any $m \in \mathcal{M}$ there is a neighbourhood $U \subset \mathcal{M}$ and a continuous (resp. Hölder) map $f : U \to C^k(D^n, \mathcal{M})$ such that for any $\tilde{m} \in U$, $f(\tilde{m})$ is a diffeomorphism of the $n$-dimensional unit disk $D^n$ onto $F_{\mathrm{loc}}(\tilde{m})$.

The foliation is called a $C^\ell$ *foliation* if for any $m$ there is a neighbourhood $U$ and a $C^\ell$ chart $\psi : U \to D^n \times D^{\dim \mathcal{M} - n}$ with $F_{\mathrm{loc}}(\psi^{-1}(0, y)) = \psi^{-1}(D^n \times \{y\})$

In the following we will be concerned with foliations which are Hölder with $C^\infty$ leaves, but in practice, it would not make a difference to us if they were only continuous.

It turns out that for $t > 0$, $\varphi_t^{A_0}$ is an example of a partially hyperbolic diffeomorphism, specifically, it is *partially hyperbolic in the narrow sense* with respect to the splitting (1.2). Partially hyperbolic diffeomorphisms are the subject of many texts, and we will refer to Pesin's book [Pes04]. In particular Section 2.2 therein contains all the relevant definitions.

By the stable manifold theorem for partially hyperbolic diffeomorphisms (see e.g. [Pes04, Theorem 4.1] we get for any $m \in \mathcal{M}$ a unique $n_s$-dimensional immersed $C^\infty$-submanifold $W^s(m) \subset \mathcal{M}$ which is tangent to the stable foliation (i.e. $T_x(W^s(m)) = E_x^s$). We call $W^s(m)$ the *stable manifold* of $m \in \mathcal{M}$ and there are $C > 0, \nu > 0$ such that it is given by

$$W^s(m) = \left\{ m' \in \mathcal{M} \,\middle|\, d_g(\varphi_t^{A_0}(m'), \varphi_t^{A_0}(m)) \leq C e^{-\nu t} \text{ for all } t > 0 \right\}. \tag{1.6}$$

It is known that the partition of $\mathcal{M}$ into stable manifolds is a Hölder foliation with $C^\infty$ leaves of the manifold $\mathcal{M}$, called the *stable foliation*. Note that by (1.6) and the commutativity of the Anosov action, we directly deduce that the foliation into stable leaves is invariant under the Anosov action, i.e. for all $a \in \mathbb{A}$, $\tau(a)(W^s(m)) = W^s(\tau(a)(m))$. This directly implies that we can define the *weak stable manifolds*

$$W^{ws}(m) = \bigcup_{a \in \mathbb{A}} W^s(\tau(a)(m)) = \bigcup_{y \in W^s(m)} \tau(\mathbb{A}) y \tag{1.7}$$

which are immersed submanifolds tangent to the neutral and stable directions , i.e. $T_x(W^{ws}(m)) = E_x^0 \oplus E_x^s$. By construction the weak stable manifolds provide again a Hölder foliation of $\mathcal{M}$ with $C^\infty$-leaves of dimension $n_{ws} = n_s + \kappa$. Precisely the same way one can define the *unstable manifolds* $W^u(m)$ and the *weak unstable manifolds* $W^{wu}(m)$ and they provide foliations with the same properties. Note that despite the fact that all foliations have $C^\infty$-leaves, none of these dynamical foliations is known to be a $C^\infty$-foliation (or even a $C^1$ foliation) in general (cf. [BFL92] for an example of what is expected to happen when one assumes smoothness of such foliations).

In order to discuss the disintegration of measures along foliations let us first introduce product neighbourhoods. We consider given $F$ and $G$ two continuous foliations with smooth leaves and assume they are complementary (i.e $TM = TF \oplus TG$). For $\delta > 0$ we denote by $B_F(m,\delta) \subset F(m)$ the ball of radius $\delta$ around $m$ inside the leaf $F(m)$. Then for any $m \in \mathcal{M}$ there is a $\delta > 0$, a neighbourhood $U$ called *product neighbourhood* (see [PS70, Theorem 3.2]) such that the following map is a homeomorphism

$$P : \begin{cases} B_F(m,\delta) \times B_G(m,\delta) & \to \quad U \\ \quad (x,y) & \mapsto G_{\mathrm{loc}}(x) \cap F_{\mathrm{loc}}(y) \end{cases}. \tag{1.8}$$

Given such a product neighbourhood $U$, we can introduce the Rokhlin disintegration of measures along $F$ in $U$.

**Proposition 1.4** (Rokhlin's theorem [Rok49])**.** *For any Borel probability measure $\mu$ on $U$ there exists a measure $\hat{\mu}$ on $B_G(m,\delta)$ and a measurable family of probability measures $\mu_y$ on $F_{\mathrm{loc}}(y)$, called conditional measures, so that for $f : U \to \mathbb{C}$ in $L^1(\mu)$,*

$$\int_U f d\mu = \int_{B_G(m,\delta)} \Big( \int_{F_{\mathrm{loc}}(y)} f(P(x,y)) d\mu_y(x) \Big) d\hat{\mu}(y). \tag{1.9}$$

*The $\mu_y$ are unique almost surely.*

The $\mu_y$ are called the conditional measures on the leaves $F_{\mathrm{loc}}(y)$. Note that by (1.9) one has that $\hat{\mu}$ is the pushforward of $\mu$ under the projection $U \cong B_F(m,\delta) \times B_G(m,\delta) \to B_G(m,\delta)$. Furthermore by the proof of Rokhlin's theorem (see e.g. the easily accessible notes by Viana [Via]) one gets a description of the conditional measures $\mu_y$. Let us therefore introduce the $F$-tubes

$$\mathcal{T}_F(y,\varepsilon) := P(B_F(m,\delta) \times B_G(y,\varepsilon)) \subset U.$$

Then for $\hat{\mu}$ almost all $y \in B_G(m,\delta)$ the limit

$$\lim_{\varepsilon \to 0} \frac{\mathbb{1}_{\mathcal{T}_F(y,\varepsilon)} \mu}{\mu(\mathcal{T}_F(y,\varepsilon))}$$

exists as a weak limit of probability measures on $U$. Obviously the limit is a probability measure supported on $F_{\mathrm{loc}}(y)$ and coincides with the conditional measures $\mu_y$ (for the points $y$ where the limit may not exist the measures $\mu_y$ can be chosen arbitrarily as they are a $\hat{\mu}$-zero set).

**Definition 1.5.** Let $F$ be a foliation on $\mathcal{M}$. We say that $F$ is *absolutely continuous* if the Riemannian volume measure $v_g$ on $\mathcal{M}$ can be disintegrated in all product neighbourhoods (with an arbitrarily chosen local smooth transversal foliation $G$ of complementary dimension) such that all the conditional measures $v_{g,y}$ are absolutely continuous with respect to the Riemannian volume measures on the local leaves $F_{\mathrm{loc}}(y)$.

We shall next denote by $L^s_x, L^u_x$ the Riemannian measure induced by restricting the metric $g$ on the local stable $W^s_{\mathrm{loc}}(x)$ and unstable $W^u_{\mathrm{loc}}(x)$ manifolds, and call it the Lebesgue measure on $W^{s/u}_{\rm loc}(x)$. Similarly we write $L^{ws}_x$, $L^{wu}_x$, $L^0_x$, for the Riemannian measure on the weak stable $W^{ws}_{\rm loc}$, weak unstable $W^{ws}_{\rm loc}(x)$ and the action direction $W^0_{\rm loc}(x) = \{\varphi^A_1(x)\,|\,|A| < \epsilon\}$, and $L^G_x$ is the Riemannian measure on $G_{\rm loc}(x)$ if $G$ is a smooth foliation.

Note that if the foliation is $C^k$ with $k > 1$ then, by Fubini's theorem, the foliation is absolutely continuous and the conditional measures have $C^{k-1}$ densities. It is worthwhile to note that if the foliation is not smooth anymore but only the leaves are, then absolute continuity does not hold in general. There are indeed examples of Hölder foliations with smooth leaves that are *not* absolutely continuous (see e.g. [Pes04, Section 7.4]). However, the stable and unstable foliations of Anosov actions are absolutely continuous. Even the following significantly stronger statement holds:

**Proposition 1.6.** *Let $X$ be a smooth Anosov action, and let $W^s, W^u$ be the associated stable and unstable foliations. Then, $W^s$ and $W^u$ are absolutely continuous in the sense of Definition 1.5. Moreover if $U \subset \mathcal{M}$ is a product neighbourhood around $m \in U$ of $W^{s/u}$ and an arbitrary smooth transversal complementary foliation $G$, and $v_g$ the Riemannian volume measure on $U$ then there is a continuous function $\delta_{W^{s/u}} : U \to \mathbb{R}^+$ such that*

$$\int_U f dv_g = \int_{G_{\rm loc}(m)} \left( \int_{W^{s/u}_{\rm loc}(y)} f(z)\delta_{W^{s/u}}(z) dL^{s/u}_y(z) \right) dL^G_m(y) \tag{1.10}$$

*Furthermore $\delta_{W^{s/u}}$ is uniformly smooth along the leaves $W^{s/u}_{\rm loc}(y)$ i.e. if $SW^{s/u}_{\rm loc}(y)$ is the sphere bundle of $W^{s/u}_{\rm loc}(y)$, then*

$$\|\delta_{W^{s/u}}\|_{C^k(W^{s/u}_{\rm loc}(y))} := \sup_{z\in W^{s/u}_{\rm loc}(y)} \sup_{X_1,\dots,X_k \in S_z W^{s/u}_{\rm loc}(y)} \left| X_1 \cdots X_k \big( (\delta_{W^{s/u}})_{|W^{s/u}_{\rm loc}(y)} \big) \right|$$

*are finite and $\|\delta_{W^{s/u}}\|_{C^k(W^{s/u}_{\rm loc}(y))}$ varies continuously in $y \in G_{\rm loc}(m)$.*

*Proof.* The absolute continuity of the stable and unstable foliation is well established in the literature (see e.g. [Pes04, Theorem 7.1] for the statement for partially hyperbolic diffeomorphisms which can once more be applied to our setting after passing to the partially hyperbolic time-one maps $\varphi^{A_0}_1$). The absolute continuity is however not enough for our purpose of using the foliations in combination with microlocal analysis. We additionally need that the $\delta_{W^{s/u}}$ are smooth along the leaves $W^{s/u}_{\rm loc}$. This smoothness seems to be folklore among dynamical systems specialists, but as the statement is not written down explicitly and is important for our further analysis, we explain how it can be deduced from existing results in the literature:

In order to simplify the notation we restrict ourselves to the case of the stable foliation $W^s$. We follow the standard approach to express the density function $\delta_{W^s}$ by holonomies and their Jacobians:

Let us consider around a point $m \in \mathcal{M}$ a local $C^\infty$-foliation $G$ that is transversal to $W^s$ and has complementary dimension $n_{wu}$. Let $U$ be a product neighbourhood of these transversal foliations. Now for any $x_1, x_2 \in W^s_{\rm loc}(m)$ we define the following *holonomy map* (cf. Figure 1) of the stable foliation

$$H^{W^s}_{x_1,x_2} \begin{cases} G_{\rm loc}(x_1) & \to G_{\rm loc}(x_2) \\ \quad z & \mapsto W^s_{\rm loc}(z) \cap G_{\rm loc}(x_2). \end{cases}$$

As the stable foliation is not smooth in general, the holonomy maps are neither. But we have

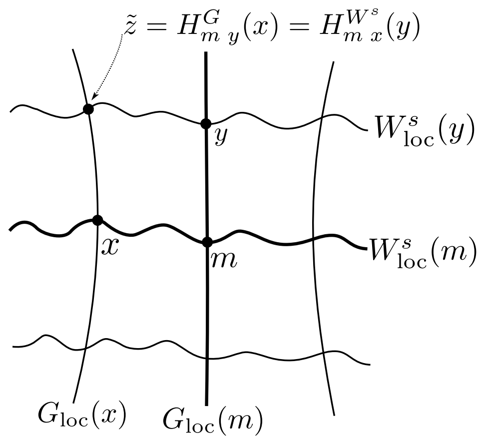


FIGURE 1. In a product neighbourhood of two complementary transversal foliations (e.g. $W^s$ and $G$ in this sketch) one can define the holonomies along the respective foliations. In the proof of Proposition 1.6 a crucial point is that one can first use a Fubini theorem w.r.t. the smooth transversal foliation $G$ and then transform this into a disintegration along $W^s$ by the use of holonomies and their Jacobians.

**Proposition 1.7** (See e.g. [Pes04, Theorem 7.1]). *The holonomy maps of the stable (and unstable) foliation are absolutely continuous, i.e. there is a measurable function $j^{W^s}_{x_1,x_2}$ on $G_{\mathrm{loc}}(x_1)$ called the Jacobian of the holonomy map such that*

$$(H^{W^s}_{x_1,x_2})_*(j^{W^s}_{x_1,x_2} L^G_{x_1}) = L^G_{x_2}$$

In the same manner on can introduce the holonomies along the foliation $G$, $H^G_{y_1,y_2} : W^s_{\mathrm{loc}}(y_1) \to W^s_{\mathrm{loc}}(y_2)$ and their Jacobians. As the foliation $G$ is smooth and the leaves $W^s$ are smooth, these holonomy maps are in fact diffeomorphisms and their Jacobians are always defined via the differential.

With the absolute continuity of $H^{W^s}_{x_1,x_2}$ one can prove that $W^s$ is absolutely continuous and give an expression for the conditional densities: first, as $G$ is a smooth foliation we use Fubini's Theorem and write

$$\int_U f dv_g = \int_{W^s_{\mathrm{loc}}(m)} \left( \int_{G_{\mathrm{loc}}(x)} f(z)\delta_G(z) dL^G_x(z) \right) dL^s_m(x)$$

with a smooth density $\delta_G \in C^\infty(U)$: Using the absolute continuity of the homeomorphism $H^{W^s}_{m,x} : G_{\mathrm{loc}}(m) \to G_{\mathrm{loc}}(x)$ we can transform the integral over $G_{\mathrm{loc}}(x)$ into an integral over $G_{\mathrm{loc}}(m)$

$$\int_U f dv_g = \int_{W^s_{\mathrm{loc}}(m)} \left( \int_{G_{\mathrm{loc}}(m)} f(H^{W^s}_{m,x}(y))\delta_G(H^{W^s}_{m,x}(y)) j^{W^s}_{m,x}(y) dL^G_m(y) \right) dL^s_m(x).$$

Finally using $H^{W^s}_{m,x}(y) = H^G_{m,y}(x) := \tilde{z}$ (cf. Figure 1) we can transform the integral over $W^s_{\mathrm{loc}}(m)$ into an integral over $W^s_{\mathrm{loc}}(y)$

$$\int_U f dv_g = \int_{G_{\mathrm{loc}}(m)} \Big( \int_{W^s_{\mathrm{loc}}(y)} f(\tilde{z})\delta_G(\tilde{z}) j^{W^s}_{m,H^G_{y,m}(\tilde{z})}(y) j^G_{y,m}(\tilde{z}) dL^s_y(\tilde{z}) \Big) dL^G_m(y) \qquad (1.11)$$

making appear the Jacobian $j^G_{y,m}$ of the holonomy map $H^G_{y,m}$. This proves the absolute continuity of the stable foliation and shows that the conditional densities on $W^s_{\mathrm{loc}}(y)$ are given by $\delta_G(\tilde z) j^{W^s}_{m,H^G_{y,m}(\tilde z)}(y) j^G_{y,m}(\tilde z)$. As $G$ was a smooth foliation $\delta_G, j^G_{y,m}$ and $H^G_{m,y}$ are smooth so it only remains to show that $j^{W^s}_{m,x}(y)$ is a smooth function in $x \in W^s_{\mathrm{loc}}(m)$ depending continuously on $y \in G_{\mathrm{loc}}(m)$. However by [Pes04, Remark 7.2] there is an explicit formula for the Jacobian for partially hyperbolic diffeomorphisms. In order to shorten the notation we introduce $\Phi := \varphi_1^{A_0}$ and we can express the Jacobian by [Pes04, (7.3)] as[5]

$$j^{W^s}_{m,x}(y) = \prod_{k=0}^{\infty} \frac{\Big| \det \Big( (d_{\Phi^k(y)}\Phi)_{|T_{\Phi^k(y)}\big(\Phi^k(G_{\mathrm{loc}}(m))\big)} \Big) \Big|}{\Big| \det \Big( (d_{\Phi^k(H^{W^s}_{m,x}(y))}\Phi)_{|T_{\Phi^k(H^{W^s}_{m,x}(y))}\big(\Phi^k(G_{\mathrm{loc}}(x))\big)} \Big) \Big|}$$

In order to analyze the regularity of this infinite product we consider the espressions $\det((d_{\Phi^k(y)}\Phi)_{|T_{\Phi^k(y)}\Phi^k(G_{\mathrm{loc}}(m))})$ as functions on the Grassmanians: Let $\mathcal{G} \to \mathcal{M}$ be the Grassmanian bundle of $n_{wu}$-dimensional subspaces in $T\mathcal{M}$. From the Riemannian metric on $\mathcal{M}$, $\mathcal{G}$ inherits a Riemannian metric[6]. Note that the map $\tilde\Phi : T_x\mathcal{M} \supset V \mapsto d\Phi_x(V) \subset T_{\Phi(x)}\mathcal{M}$ is a canonical lift of the diffeomorphism $\Phi$ to the Grassmanian bundle. Furthermore we can define $\mathcal{J} : \mathcal{G} \ni (x,V) \mapsto |\det((d_x\Phi)_{|V})| \in \mathbb{R}_{>0}$ which is a smooth function. We can thus write

$$\log(j^{W^s}_{m,x}(y)) = \sum_{k=0}^{\infty} \log \mathcal{J}(\tilde\Phi^k(T_y G_{\mathrm{loc}}(m))) - \log \mathcal{J}(\tilde\Phi^k(T_{H^{W^s}_{m,x}(y)} G_{\mathrm{loc}}(x))). \tag{1.12}$$

We note that by definition of the holonomies, $y$ and $H^{W^s}_{m,x}(y)$ lie on the same stable manifold $W^s_{\mathrm{loc}}(y)$ and the subspaces $S(m) := T_y G_{\mathrm{loc}}(m)$ and $S(x) := T_{H^{W^s}_{m,x}(y)} G_{\mathrm{loc}}(x)$ are both transversal and complementary to $W^s_{\mathrm{loc}}(y)$. An immediate consequence of the partial hyperbolicity of $\Phi$ is that those two spaces (considered as points in the Grassmanian) become exponentially close under the lifted action of $\tilde\Phi$, i.e. there are $C, \nu > 0$ such that

$$d_{\mathcal{G}}\Big(\tilde\Phi^k(S(m)), \tilde\Phi^k(S(x))\Big) \le C e^{-\nu k} d_{\mathcal{G}}\Big(S(m), S(x)\Big). \tag{1.13}$$

Now the compactness of $\mathcal{G}$ implies that $\mathcal{J}$ is uniformly Lipshitz and thus the series in (1.12) converges absolutely which implies that $j^{W^s}_{m,x}(y)$ is a continuous function (in the $y$ variable as well as in the $x$ variable). We now show that it is even differentiable w.r.t. $x$: let us therefore take a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to W_s(m)$, $\gamma(0) = x$ and $\|\gamma'(t)\| \le 1$ then

$$\frac{d}{dt}_{|t=0} \log j^{W^s}_{m,\gamma(t)}(y) = -\sum_{k=0}^{\infty} d(\log \mathcal{J})\big[\frac{d}{dt}_{|t=0} \tilde\Phi^k(S(\gamma(t)))\big]. \tag{1.14}$$

Using once more the hyperbolicity (in the form of (1.13)) we obtain the estimate

$$\left\| \frac{d}{dt}_{|t=0} \tilde\Phi^k(S(\gamma(t))) \right\|_{T_{S(x)}\mathcal{G}} \le C' e^{-\nu k}$$

and this time the uniform Lipshitzness of $d \log \mathcal{J}$ ensures the absolute convergence of the right hand side of (1.14) which implies that $\frac{d}{dt}_{|t=0} \log j^{W^s}_{m,\gamma(t)}(y)$ exists and its value depends

[5]Be aware that Pesin uses the inverted diffeomorphism $\Phi^{-1}$ in his formula but the numerator and denominator in his formula are also interchanged so that the formula agrees with the one that we use here.

[6]There might be different choices of metrics on the Grassmanian-fibres, but they all lead to equivalent metrics.

continuously on $y$. Iteratively we obtain the same statement for higher derivatives. Indeed, we have that the $n$-th derivative of $\tilde{\Phi}^k(T_{H^{W^s}_{m,x}(y)}G_{\mathrm{loc}}(x))$ with respect to $x \in W_s(m)$ decays exponentially in $k$, this can be checked by writting $S(x) := T_{H^{W^s}_{m,x}(y)}G_{\mathrm{loc}}(x)$ and by induction on $n$. Up to possibly multiplying $A_0$ by a fixed positive integer, we can assume that $\|d\tilde{\Phi}(\tilde{\Phi}^j(S(x)))\| \le e^{-\nu+\varepsilon}$ for all $j \ge 0$ and $\epsilon > 0$ small by the discussion above. For $n = 2$, write

$$\begin{aligned} d(\tilde{\Phi}^k(S(x))) &= d\tilde{\Phi}(\tilde{\Phi}^{k-1}(S(x)))d\tilde{\Phi}(\tilde{\Phi}^{k-2}(S(x)))\dots d\tilde{\Phi}(S(x))dS(x), \\ d^2(\tilde{\Phi}^k(S(x))) &= \sum_{j=1}^{k-1} d\tilde{\Phi}(\tilde{\Phi}^{k-1}(S(x)))\cdots d^2\tilde{\Phi}(\tilde{\Phi}^j(S(x))d(\tilde{\Phi}^j(S(x)))\dots d\tilde{\Phi}(S(x))dS(x) \\ &\quad + d\tilde{\Phi}(\tilde{\Phi}^{k-1}(S(x)))\dots d\tilde{\Phi}(S(x))d^2S(x). \end{aligned}$$

This implies, using that $d^2\tilde{\Phi}$ is uniformly bounded on $\mathcal{G}$, that

$$\|d^2(\tilde{\Phi}^k(S(x)))\| \le Cke^{-(\nu-\varepsilon)(k-1)}.$$

Then repeating the same argument with an easy induction on $n$ gives the result for derivatives of order $n$ (we refer to the proof of [dlL92, Lemma 5.5] for example for more details). Combining this estimates with (1.11) we obtain the desired smoothness of $\delta_{W^s}$ along the leaves $W^s_{\mathrm{loc}}$. □

Note that the proof of Proposition 1.6 strongly depends on the fact that there is an exponential contraction along the stable, resp. unstable manifold and it would fail when working directly on the weak-(un)stable foliation. Nevertheless thanks to the fact that the neutral foliation is a smooth foliation we can establish the same result as Proposition 1.6. We will do this in two steps: In a first step we show that there is a continuous density function $\delta_{W^{ws}}$ for the weak-stable foliations[7] and give an explicit expression in terms of $\delta_{W^s}$ and some further quantitities which we introduce now: by (1.7) we have

$$W^{ws/wu}_{\mathrm{loc}}(m) = \bigcup_{y\in W^{s/u}_{\mathrm{loc}}(m)} W^0_{\mathrm{loc}}(m) = \bigcup_{x\in W^0_{\mathrm{loc}}(m)} W^{s/u}_{\mathrm{loc}}(x), \tag{1.15}$$

where $W^0_{\mathrm{loc}}(y) := (\tau(\mathbb{A})y) \cap W^{ws/wu}_{\mathrm{loc}}(m)$ is simply the $\mathbb{A}$-orbit through $y$. By the fact that $W^{s/u}(\tau(a)m) = \tau(a)(W^{s/u}(m))$ and the smoothness of the Anosov action both partitions of the leaf $W^{ws/wu}_{\mathrm{loc}}(m)$ are smooth foliations of $W^{ws/wu}_{\mathrm{loc}}(m)$ and by Fubini there are strictly positive, smooth functions $\delta^{W^{ws/wu}_{\mathrm{loc}}(m)}_{W^0} \in C^\infty(W^{ws/wu}_{\mathrm{loc}}(m))$ and $\delta^{W^{ws/wu}_{\mathrm{loc}}(m)}_{W^{s/u}} \in C^\infty(W^{ws/Wu}_{\mathrm{loc}}(m))$ such that

$$\begin{aligned} &\int_{W^{ws/wu}_{\mathrm{loc}}(m)} f(y)dL^{ws/wu}_m(y) \\ &\qquad = \int_{W^{s/u}_{\mathrm{loc}}(m)} \left(\int_{W^0_{\mathrm{loc}}(y')} f(z)\delta^{W^{ws/wu}_{\mathrm{loc}}(m)}_{W^0}(z)dL^0_{y'}(z)\right) dL^{s/u}_m(y') && (1.16) \\ &\qquad = \int_{W^0_{\mathrm{loc}}(m)} \left(\int_{W^{s/u}_{\mathrm{loc}}(y')} f(z)\delta^{W^{ws/wu}_{\mathrm{loc}}(m)}_{W^{s/u}}(z)dL^{s/u}_{y'}(z)\right) dL^0_m(y'). && (1.17) \end{aligned}$$

[7]Again the case of weak unstable foliation follows exactly the same way but we only focus on thw weak stable case to simplify the notation.

Now, in the proof of Proposition 1.6 we chose a transversal complementary smooth foliation $G$ such that $G_{\mathrm{loc}}(m) = W^{wu}_{\mathrm{loc}}(m)$. With (1.10), (1.16) and (1.17) this yields

$$\int_U f\, dv_g = \int_{W^u_{\mathrm{loc}}(m)} \left( \int_{W^{ws}_{\mathrm{loc}}(y)} f(z)\delta_{W^{ws}}(z) dL^{ws}_y(z) \right) dL^u_m(y). \tag{1.18}$$

with

$$\delta_{W^{ws}}(z) = \frac{\delta_{W^s}(z)\delta^{W^{wu}_{\mathrm{loc}}(m)}_{W^0}\big(\mathrm{pr}_{W^{ws}_{\mathrm{loc}}(y)\to W^0_{\mathrm{loc}}(y)}(z)\big)}{\delta^{W^{ws}_{\mathrm{loc}}(y)}_{W^s}(z)}$$

Here $\mathrm{pr}_{W^{ws}_{\mathrm{loc}}(y)\to W^0_{\mathrm{loc}}(y)}$ is the projection along the smooth subfoliation (1.15) of $W^{ws}_{\mathrm{loc}}(y)$. In order to obtain an analogue to Proposition 1.6 it remains to analyze the regularity of $\delta_{W^{ws}}$ along the leaves $W^{ws}_{\mathrm{loc}}(y)$. Note that for fixed $y$ by the smoothness of the subfoliations (1.15) of $W^{ws}_{\mathrm{loc}}(y)$ we conclude that the functions $\delta^{W^{ws}_{\mathrm{loc}}(y)}_{W^s}(z)$ and $\delta^{W^{wu}_{\mathrm{loc}}(m)}_{W^0}\big(\mathrm{pr}_{W^{ws}_{\mathrm{loc}}(y)\to W^0_{\mathrm{loc}}(y)}(z)\big)$ are smooth on $W^{ws}_{\mathrm{loc}}(y)$. By the Hölder continuity of the weak stable foliation their $C^k_{W^{ws}_{\mathrm{loc}}(y)}$-norms vary continuously on $y \in W^u_{\mathrm{loc}}(m)$. However, for $\delta_{W^s}(z)$ we only know so far that this density is smooth along $W^s_{\mathrm{loc}}(y)$. Using the smoothness of the Anosov action we can improve this further: for any $z \in W^s_{\mathrm{loc}}(y) \subset U$ and any $a \in V \subset \mathbb{A}$ where $V$ is a neighbourhood of the identity such that $\tau(V)z \subset U$ we get the following equivariance property which can be derived from a straightforward calculations using the $\mathbb{A}$ invariance of the weak-(un)stable foliations as well as several occurences of the transformation formula:

$$\delta_{W^s}(\tau(a)z) = \frac{|\det(d\tau(a))|(z)}{|\det(d\tau(a)_{|E^{wu}})|(y)\cdot |\det(d\tau(a)_{|E^s})(z)|}\delta_{W^s}(z).$$

All the appearing Jacobians here are understood with respect to the respective Riemannian volume measures. As the Jacobians depend smoothly on $a$ this shows that $\delta_{W^s}$ has also bounded derivatives of any order into the direction of the $\mathbb{A}$-orbits. Summarizing we have shown:

**Proposition 1.8.** *Precisely the same statement as Proposition 1.6 holds when replacing the (un)stable foliation $W^{s/u}$ by the weak (un)stable foliation $W^{ws/wu}$.*

As a consequence of Proposition 1.8 we can prove the following crucial result which is a slightly more general version of [Wei17, Prop 6] for Anosov actions. It connects classical regularity of functions into the directions of a dynamical Hölder foliations with its microlocal regularity i.e. the wavefront set:

**Lemma 1.9.** *Let $\tau$ be an $\mathbb{R}^\kappa$-Anosov action, and consider its weak-unstable foliation. Let $f$ be a measurable function on $\mathcal{M}$, such that for Lebesgue almost all $p \in \mathcal{M}$, $f_{|W^{wu}_{\mathrm{loc}}(p)}$ is smooth with all derivatives along the leaves being uniformly bounded with respect to $p$. Then $\mathrm{WF}(f) \subset E^*_u$.*

*Proof.* We pick a point $p \in M$, $\xi \in T^*_p\mathcal{M}$, such that $\xi \notin E^*_u$ and a smooth function $S$ such that $d_pS = \xi$. Let $G$ be a transverse foliation to $W^{wu}$ near $p$, and we can assume for example that $G(p) = W^s(p)$. Then, using Proposition 1.6 or more particularly (1.18) (with the weak-unstable foliation instead of the weak-stable foliation), for each $\chi \in C^\infty(\mathcal{M})$ supported in a small neighbourhood of $p$

$$\Big|\int \chi e^{\frac{i}{h}S} f dv_g\Big| \le \int_{W^s(p)} \left| \int_{W^{wu}_{\mathrm{loc}}(y)} \chi(x)e^{\frac{i}{h}S(x)} f(x)\delta_{W^{wu}(y)}(x) dL^{wu}_y(x)\right| dL^s_p(y).$$

Since each $W^{wu}_{\rm loc}(y)$ is a smooth manifold, and $f$ restricted to this manifold is smooth, we can integrate by parts in the variable $x$. Here, it is crucial that $\delta_{W^{wu}(y)}(x)$ is smooth in $x$. We deduce (since the estimate on $f$ are locally uniform) that this integral is $\mathcal{O}(h^\infty)$, as soon as $dS_{|W^{wu}_{loc}(p)}$ does not vanish. But the condition that $\xi \notin E_u^*$ ensures this close enough to $p$, since $E_u^*$ is exactly the set of covectors that vanish on $E^u \oplus \mathbb{R}X$. So for $\chi$ supported close to $p$ one gets the desired result. This implies that $\xi \notin \mathrm{WF}(f)$. □

1.3. **Invariant measures via spectral theory.** In this section we state the results about the physical measures for general Anosov action as they have been obtained in [GBGHW20] and we also recall the essential constructions on which our analysis will be based.

The existence was obtained through the theory of *Ruelle-Taylor resonances*, which are defined as a joint spectrum of the family of vector fields $X_A$ for $A \in \mathfrak{a}$ in certain functional spaces. More precisely, we say that $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is a Ruelle-Taylor resonance for $\tau$ if and only if there exists $u \in C^{-\infty}(\mathcal{M})$ non-zero with $\mathrm{WF}(u) \subset E_u^*$ (here WF denotes Hörmander wave-front set of the distribution [Hör03, Chapter 8]) and

$$\forall A \in \mathfrak{a}, \quad -X_A u = \lambda(A)u. \tag{1.19}$$

We say that $u$ is a *joint Ruelle resonant state* of $X$. Using the operator $d_X : C^{-\infty}(\mathcal{M}) \to C^\infty(\mathcal{M}) \otimes \mathfrak{a}^*$ defined by $(d_X u)(A) := X_A u$ for all $A \in \mathfrak{a}$, the system (1.19) can be rewritten under the form $(-d_X - \lambda)u = 0$.

Given a general Anosov action $X$, we choose vectors $A_1, \dots, A_\kappa$ in the Weyl chamber $\mathcal{W}$, which form a basis of $\mathfrak{a}$. The dual basis in $\mathfrak{a}^*$ is denoted $(e_j)_j$, and set $X_j := X_{A_j}$, and we use $dv_g$ the smooth Riemannian probability measure on $\mathcal{M}$. If we further pick a non-negative function $\psi_j \in C_c^\infty(\mathbb{R}^+)$ satisfying $\int_0^\infty \psi_j(t)dt = 1$ then we can consider the operator

$$R := \prod_{j=1}^{\kappa} \int e^{-t_j X_j} \psi_j(t_j) dt_j. \tag{1.20}$$

This operator appeared in a parametrix construction in a Taylor complex generated by the Anosov action and this parametrix was the central ingredient for establishing the existence of the Ruelle-Taylor resonances in [GBGHW20]. For the purpose of this paper we will not need to introduce the Taylor complexes and spectrum but we will only focus on the results needed for our present work. In section 4.1 of [GBGHW20], we construct a function $G \in C^\infty(T^*\mathcal{M})$, called *escape function for* $A_1 \in \mathcal{W}$ and satisfying the properties of [GBGHW20, Definition 4.1]: in particular, there is $R_0 > 0$, $c_X > 0$ and a conic neighborhood $\Gamma_{E_0^*}$ of $E_0^*$ in $T^*\mathcal{M}$ such that

$$\begin{cases} G(x,\xi) = m(x,\xi)\log(1+f(x,\xi)), \\ f \in C^\infty(T^*\mathcal{M}, \mathbb{R}^+) \text{ positive and homogeneous of degree 1 in } |\xi| > R_0, \\ m \in C^\infty(T^*\mathcal{M}, [-1/2, 8]) \text{ homogeneous of degree 0 in } |\xi| > R_0, \\ m \le -1/4 \text{ in an arbitrarily small conic neighborhood } \Gamma_u \text{ of } E_u^* \text{ if } |\xi| > R_0 \\ m \ge 1/2 \text{ outside an arbitrarily small conic neighborhood } \Gamma_u' \text{ of } \Gamma_u \text{ if } |\xi| > R_0. \\ (\bigcup_{t\in[0,1]} e^{tX_A^H}(x,\xi)) \cap (\Gamma_{E_0^*} \cup \{|\xi| < R_0\}) = \emptyset \Rightarrow G(e^{X_A^H}(x,\xi)) - G(x,\xi) \le -c_X \end{cases} \tag{1.21}$$

where $X_A^H$ is the Hamilton vector field of the principal symbol $p := \xi(X_A(x))$ of the operator $-iX_A$ (we note that its flow $e^{tX_A^H}$ is the symplectic lift of $\varphi_1^A$).

After fixing a quantization procedure Op mapping symbols on $T^*\mathcal{M}$ to operators acting on $C^\infty(\mathcal{M})$ (as in [Zwo12]), we consider the pseudo-differential operator $\mathrm{Op}(e^{NG})$ with variable order and we define the Hilbert space

$$\mathcal{H}^{NG} := \mathrm{Op}(e^{NG})^{-1}L^2(\mathcal{M}).$$

where $\mathrm{Op}(e^{NG})$ can be made invertible by choosing appropriately $G$. For later, we will also need a semi-classical parameter $h \in (0,1]$, to consider a semiclassical quantization $\mathrm{Op}_h$ and to define

$$\mathcal{H}^{NG}_h := (\mathrm{Op}_h(e^{NG}))^{-1}L^2(\mathcal{M}).$$

The spaces $\mathcal{H}^{NG}_h$ for different values of $h$ are the same topological vector spaces but the norms are different. For more details on the construction of the anisotropic spaces and the used microlocal techniques we refer to [GBGHW20, Section 4.1 and Appendix A].

**Proposition 1.10** ([GBGHW20, Lemma 4.14, Lemma 5.1 and Lemma 5.2])**.** *For $N > 0$ large enough, the operator $R$ of (1.20) is a bounded operator on $\mathcal{H}^{NG}$ with essential spectrum contained in the disk $D(0, 1/2)$. The only eigenvalue $s$ with $|s| = 1$ is $s = 1$ and this eigenvalue has finite multiplicity and no Jordan blocks. Finally, if $\Pi$ denotes the finite rank spectral projector of $R$ at $s = 1$, then the following convergence holds in $\mathcal{L}(\mathcal{H}^{NG})$*

$$\lim_{k\to+\infty} R^k = \Pi.$$

We note that the same results hold on the spaces $\mathcal{H}^{NG}_h$ for all $h \in (0,1)$. We will also need the technical Lemma (see [GBGHW20, Proof of Lemma 4.12]) which follows from the flexibility of the choice of the function $G$:

**Lemma 1.11.** *A distribution $u \in \mathcal{C}^{-\infty}(\mathcal{M})$ having $\mathrm{WF}(u) \subset E_u^*$ satisfies $u \in \mathcal{H}^{NG}$ for some $N > 0$. Conversely, if there is $N_0$ such that $u \in \mathcal{H}^{NG}$ for all $N \geq N_0$ and all admissible escape functions $G$ (in the sense of [GBGHW20, Definition 4.1]), then $\mathrm{WF}(u) \subset E_u^*$.*

If $F \subset T^*\mathcal{M}$ is a conical closed set in $T^*\mathcal{M}$, we denote by $C^{-\infty}_F(\mathcal{M}) := \{u \in C^{-\infty}(\mathcal{M}) \,|\, \mathrm{WF}(u) \subset F\}$.

The spectral projector $\Pi$ satisfies $R\Pi = \Pi = \Pi R$, and by [GBGHW20, Lemma 5.3] its Schwartz kernel is independent of $N, G$ and has the form

$$\Pi = \sum_{j=1}^{r} v_j \otimes \omega_j^* \tag{1.22}$$

with $v_j \in \mathcal{H}^{NG} \cap C^{-\infty}_{E_u^*}(\mathcal{M})$ and $w_j^* \in (\mathcal{H}^{NG})^* \cap C^{-\infty}_{E_s^*}(\mathcal{M})$; moreover if $N > 0$ is large enough we have by [GBGHW20, Lemma 5.3]

$$\mathrm{ran}\,\Pi = \{u \in C^{-\infty}_{E_u^*}(\mathcal{M}) \,|\, \forall A \in \mathfrak{a},\, X_A u = 0\} = \{u \in \mathcal{H}^{NG} \,|\, \forall A \in \mathfrak{a},\, X_A u = 0\}. \tag{1.23}$$

The relation of $\Pi$ with the physical measure is explained by [GBGHW20, Proposition 5.4] as follows:

**Proposition 1.12.**

*(1) For each $v \in C^\infty(\mathcal{M}, \mathbb{R}^+)$, the map*

$$\mu_v : u \in C^\infty(\mathcal{M}) \mapsto \langle \Pi u, v\rangle$$

*is a Radon measure with mass $\mu_v(\mathcal{M}) = \int_{\mathcal{M}} v\, dv_g$, invariant by $X_j$ for all $j = 1, \dots, \kappa$ in the sense $\mu_v(X_j u) = 0$ for all $u \in C^\infty(\mathcal{M})$. The space*

$$\mathrm{span}\{\mu_v \,|\, v \in C^\infty(\mathcal{M})\} = \Pi^*(C^\infty(\mathcal{M}))$$

*is, for $N$ sufficiently large, a finite dimensional subspace of $(\mathcal{H}^{NG})^*$ whose dimension equals the space of joint resonant states. Furthermore $\Pi(C^\infty(\mathcal{M}))$ is precisely spanned by the invariant measures $\mu$ with $WF(\mu) \subset E_s^*$.*

*(2) For each open cone $\mathcal{C} \subset \mathcal{W}$ in the positive Weyl chamber with $\partial\mathcal{C} \cap \partial\mathcal{W} = \{0\}$, $e_1 \in \mathfrak{a}^*$ so that $e_1 > 0$ on $\mathcal{C}$, and $u, v \in C^\infty(\mathcal{M})$,*

$$\mu_v(u) = \lim_{T\to\infty} \frac{1}{|\mathcal{C}_T|} \int_{A\in\mathcal{C}_T,} \langle \varphi_{-1}^A u, v\rangle dA \tag{1.24}$$

*where $dA$ is the Lebesgue-Haar measure on $\mathfrak{a}$ and $\mathcal{C}_T := \{A \in \mathcal{C} \,|\, e_1(A) \in (0,T)\}$. In particular, $\mu_1$ is the physical measure.*

*(3) Let $v_1, v_2 \in C^\infty(\mathcal{M}, \mathbb{R}^+)$ with $v_1 \leq C v_2$ for some $C > 0$. Then $\mu_{v_1}$ is absolutely continuous with bounded density with respect to $\mu_{v_2}$. In particular any $\mu_v$ is absolutely continuous with respect to $\mu_1$.*

*(4) Consider a local stable manifold $W^s_{\mathrm{loc}}(x_0)$, $f \in C_c^\infty(W^s_{\mathrm{loc}}(x_0), \mathbb{R}^+)$ and $L^s_{x_0}$ the Lebesgue measure on $W^s_{\mathrm{loc}}(x_0)$ with $\int f dL^s_{x_0} = 1$. Then for $\mathcal{C}$, $\mathcal{C}_T$ as in (2), the limit*

$$\mu_{fL^s_{x_0}}(u) = \lim_{T\to\infty} \frac{1}{|\mathcal{C}_T|} \int_{\mathcal{C}_T} \int_{\mathcal{M}} u(\varphi_{-1}^A) f dL^s_{x_0} dA \tag{1.25}$$

*exists and defines a probability measure in $\Pi^*(C^\infty(\mathcal{M}))$.*

*Remark* 1.13. We will see in Section 2 that in fact the whole space $\Pi^*(C^\infty(\mathcal{M}))$ is spanned by measures of the form $\mu_{fL^s_{x_0}}$.

*Proof.* The items (1), (2), (3) are proved in [GBGHW20, Proposition 5.4]. Let us prove the 4th item. For this, we follow closely the proof of [GBGHW20, Proposition 5.4] but we replace the smooth function $v$ by the measure $v' := fL^s_{x_0}$. Since $W^s_{\mathrm{loc}}(x_0)$ is a smooth submanifold of $\mathcal{M}$ we have that (see for instance [Hör03, Example 8.2.5]) $\mathrm{WF}(fL^s_{x_0}) \subset E_s^* \oplus E_0^*$. The distribution $v'$ belongs to $(\mathcal{H}^{NG})^*$: indeed for $u \in C^\infty(\mathcal{M})$,

$$\langle u, v'\rangle_{C^\infty, C^{-\infty}} = \langle \mathrm{Op}(e^{NG})u, (\mathrm{Op}(e^{NG})^{-1})^* v'\rangle_{C^\infty, C^{-\infty}}$$

and $(\mathrm{Op}(e^{NG})^{-1})^*$ is a pseudo-differential operator with variable order and with principal symbol $e^{-NG} \leq C|\xi|^{-N/2}$ in $\{(x,\xi) \in T^*\mathcal{M} \,|\, \xi \notin \Gamma_u', |\xi| > R_0\}$ by (1.21); since $\Gamma_u' \cap (E_0^* \oplus E_s^*) = \emptyset$, we deduce that there is $N > 0$ large enough so that $(\mathrm{Op}(e^{NG})^{-1})^* v' \in L^2$ (note that $v' \in H^{-\dim(\mathcal{M}+1)/2}(\mathcal{M})$). Recalling that $\mathrm{Op}(e^{NG}) : \mathcal{H}^{NG} \to L^2$ is an isometry we see that $u \mapsto \langle u, v\rangle_{C^\infty, C^{-\infty}}$ extends to a continuous functional on $\mathcal{H}^{NG}$

For $u \in C^\infty(\mathcal{M})$ we thus have using Proposition 1.10 (notice that the result is independent of the choice of $\psi$ in the definition of the operator $R$)

$$\lim_{k\to\infty} \langle R^k u, v'\rangle_{\mathcal{H}^{NG}, (\mathcal{H}^{NG})^*} = \langle \Pi u, v'\rangle_{\mathcal{H}^{NG}, (\mathcal{H}^{NG})^*} = \sum_{j=1}^{r} \omega_j^*(u) \langle v_j, v'\rangle_{\mathcal{H}^{NG}, (\mathcal{H}^{NG})^*} \tag{1.26}$$

which proves that $\mu^k_{v'} : u \mapsto \langle R^k u, v'\rangle_{\mathcal{H}^{NG}, (\mathcal{H}^{NG})^*}$ is a Radon measure that converges to some element in $\Pi^*(C^\infty)$ depending only on $v'$. The limit is a probability measure since, using $R\,1 = 1$, one has $\mu^k_{v'}(1) = \langle 1, v'\rangle = 1$. Next, in the proof of Proposition 5.4 of [GBGHW20], we replaced the functions $\prod_{j=1}^{\kappa} \psi_j(t_j)$ by $\psi_\sigma(t) = \prod_{j=1}^{\kappa} \psi_j(t_j - \sigma_j)$ for $\sigma \in \mathbb{R}^\kappa \simeq \mathfrak{a}$ small in the definition of $R^k$, and call $R^k_\sigma$ the resulting operator. Fixing one direction $A_1 \in \mathcal{C}$, $e_1 \in \mathfrak{a}^*$ so that $e_1(A_1) = 1$, and taking a transverse hypersurface $\Sigma = e_1^{-1}(\{1\})$ to $\mathcal{C}$, we obtain coordinates $(t_1, \dots, t_\kappa)$ with $t_1 = e_1(A)$ on $\mathfrak{a}$ and $\bar{t} = (t_2, \dots, t_\kappa)$ some linear coordinates on

$\Sigma$ associated with a basis $A_2, \dots A_\kappa$ of $\ker e_1$. We prove in [GBGHW20, Lemma 5.5] that if $\omega \in C_c^\infty((0,1))$ satisfies $\int \omega = 1$ and $q \in C_c^\infty(\Sigma \cap \mathcal{C})$ satisfies $\int q = 1$, then for each $v \in C^\infty(\mathcal{M})$

$$\Big|\frac{1}{N}\sum_{k=1}^{N} \omega(\tfrac{k}{N})\langle R^k_{\sigma(\bar t)}u, v\rangle q(\bar t)d\bar t - \frac{1}{N^\kappa}\int_0^N\int_{\mathbb{R}^{\kappa-1}} \langle e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}}u, v\rangle (\tfrac{t_1}{N})^{1-\kappa}\omega(\tfrac{t_1}{N})q(\tfrac{\bar t}{t_1})dt_1 d\bar t\Big|$$
$$\leq \epsilon(N)\sup_{t_1,\bar t} |\langle e^{-\sum_{j=1}^{\kappa} t_j X_{A_j}}u, v\rangle| \leq \epsilon(N)\|u\|_{C^0(\mathcal{M})}\|v\|_{(C^0(\mathcal{M}))^*}$$

where $\sigma(\bar t) = (1, \bar t)$ in the coordinates $t_1, \dots, t_\kappa$ and $\epsilon(N) \to 0$ as $N \to \infty$. We can then take a sequence $v_n \to v'$ in the topology $(C^0(\mathcal{M}))^*$ dual to $C^0(\mathcal{M})$ (i.e. Radon measures) and we obtain, using (1.26) and letting $N \to \infty$,

$$\langle u, \Pi v'\rangle = \lim_{N\to\infty} \frac{1}{N^\kappa}\int_0^N\int_{\mathbb{R}^{\kappa-1}} \langle e^{-\sum_{j=1}^{n} t_j X_{A_j}}u, v'\rangle (\tfrac{t_1}{N})^{1-\kappa}\omega(\tfrac{t_1}{N})q(\tfrac{\bar t}{t_1})dt_1 d\bar t.$$

By finally letting $\omega(t_1)q(\bar t/t_1)$ be arbitrarily close to $t_1^{\kappa-1}\mathbf{1}_{[0,1]}(t_1)\mathbf{1}_{\Sigma\cap\mathcal{C}}(\bar t/t_1)$ in $L^\infty$, we obtain that $\mu_{fL^s_{x_0}}$ can be written as (1.25). ☐

Now, using the usual proof for the decomposition of the SRB measures along stable leaves in rank 1, we can prove the

**Proposition 1.14.** *For each $x_0$ and $f \in C_c^\infty(W^s_{\rm loc}(x_0), \mathbb{R}^+)$ with $\int f dL^s_{x_0} = 1$, the measure $\mu_{fL^s_{x_0}}$ defined by (1.25) has strictly positive smooth conditional measures on the local leaves $W^s_{\rm loc}(x)$ with respect to Lebesgue measure $L^s_x$, for each $x \in \mathcal{M}$.*

*Proof.* We follow the proof of [You95, Theorem 6.3.1]. Call $\nu_0 := fL^s_{x_0}$ and $\nu_A = (\varphi^A_{-1})_*\nu_0$ for $A \in \mathcal{W}$. This measure is supported on the piece of stable manifold $\varphi^A_{-1}(W^s_{\rm loc}(x_0))$. We also define $\nu^T := \frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\nu_A dA$. Let $x_1 \in \mathcal{M}$ and consider a small neighborhood $U_{x_1}$ such that the disintegration along stable leaves can be done by Proposition 1.4. Let $y \in U_{x_1}$ be such that $\varphi^A_1(y) \in W^s_{\rm loc}(x_0)$ then the measure $\nu_A$ on a stable leaf $W^s_{\rm loc}(y) \subset U_{x_1}$ can be written as

$$\nu_A = |\det(d\varphi^A_1|_{E_s})| f \circ \varphi^A_1\, L^s_y$$

and $|\det(d\varphi^A_1|_{E_s})|$ is the Jacobian restricted to $E_s$ (computed w.r.t. Lebesgue), which is exponentially small as $|A|$ is large. Then, let $\nu^\infty := \mu_{fL^s_{x_0}} = \lim_{T\to\infty}\nu^T$ that we decompose in $U_{x_1}$ along stable leaves and we call $\nu^\infty_{x_1}$ the conditional on $W^s_{\rm loc}(x_1)$. If $G$ is a transverse smooth foliation manifold to the stable foliation near $x_1$, one can write

$$\nu^\infty_{x_1} = \lim_{\epsilon\to 0}\frac{\mathbf{1}_{F_\epsilon}\nu^\infty}{\nu^\infty(F_\epsilon)}$$

where $F_\epsilon = \cup_{y\in B_G(x_1,\epsilon)} W^s_{\rm loc}(y)$ is a tube of radius $\epsilon > 0$ around $W^s_{\rm loc}(x_1)$, where $B_G(x_1,\epsilon) \subset G(x_1)$ is the ball of radius $\epsilon$ in $G(x_1)$. Now let $V \subset W^s_{\rm loc}(x_1)$ be a small open ball and consider $\mathcal{V}_\epsilon := \cup_{y\in V} G(y)\cap F_\epsilon$. For $|A|$ large, the leaf $\varphi^A_{-1}(W^s_{\rm loc}(x_0))$ becomes long and intersect $\mathcal{V}_\epsilon$ into, say $k_\epsilon(A) \in \mathbb{N}\cup\{+\infty\}$ connected components[8] $(W^s_{\rm loc}(y_j)\cap\mathcal{V}_\epsilon)_{j=1,\dots,k_\epsilon(A)}$ for $y_j(A) \in B_G(x_1,\epsilon)$;

[8]The number of components with non-zero Lebesgue measure is countable.

note that $y_j(A)$ depends continuously on $A$. We then write

$$\begin{aligned}\nu^T(\mathcal{V}_\epsilon) &= \frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\nu_A(\mathcal{V}_\epsilon)dA \\ &= \frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\int_{W^s_{\mathrm{loc}}(y_j(A))\cap\mathcal{V}_\epsilon}|\det(d\varphi_1^A(z)|_{E_s(z)})|f(\varphi_1^A(z))\,dL^s_{y_j(A)}(z)dA.\end{aligned} \tag{1.27}$$

We can now use the holonomy map $H^G_{x_1,y_j(A)}$ as defined after Proposition 1.7 to rewrite the $W^s_{\mathrm{loc}}(y_j(A))$ integral as a $W^s_{\mathrm{loc}}(x_1)$ integral:

$$\begin{aligned}&\int_{W^s_{\mathrm{loc}}(y_j)\cap\mathcal{V}_\epsilon}|\det(d\varphi_1^A(z)|_{E_s(z)})|f(\varphi_1^A(z))\,dL^s_{y_j}(z) \\ &= \int_V|\det(d\varphi_1^A(H^G_{x_1,y_j(A)}(z))|_{E_s})|f(\varphi_1^A(H^G_{x_1,y_j(A)}(z)))j^G_{x_1,y_j(A)}(z)\,dL^s_{x_1}(z)\end{aligned}$$

where $j^G_{x_1,y_j}$ is the Jacobian of $H^G_{x_1,y_j}$ which is uniformly bounded above and below (with respect to $y_j$) by some positive constants $C_1, C_2$. Let us define the density

$$\rho_j^A(z) = |\det(d\varphi_1^A(H^G_{x_1,y_j(A)}(z))|_{E_s})|f(\varphi_1^A(H^G_{x_1,y_j(A)}(z)))j^G_{x_1,y_j(A)}(z)$$

on a neighborhood of $V$ inside $W^s_{\mathrm{loc}}(x_1)$. Now we can show that there is $C > 0$ such that for all $y, z$ in $W^s_{\mathrm{loc}}(x_1)\cap V$, all $j \le k_\epsilon(A)$ and all $A \in \mathcal{C}$

$$\frac{\rho_j^A(z)}{\rho_j^A(y)} \le e^{Cd_g(y,z)}. \tag{1.28}$$

To obtain this estimate we can apply the argument in [You95, Proof of Theorem 5.2.1]: First, since $H^G_{x_1,y_j}(z), H^G_{x_1,y_j}(y)$ are on the same the stable leaf, there is a uniform $C > 0$ such that for all $A \in \mathcal{C}_T$ and $j \le k_\epsilon(A)$ and all $y, z \in W^s_{\mathrm{loc}}(x_1)\cap V$

$$\frac{f(\varphi_1^A(H^G_{x_1,y_j(A)}(y)))j^G_{x_1,y_j(A)}(y)}{f(\varphi_1^A(H^G_{x_1,y_j(A)}(z)))j^G_{x_1,y_j(A)}(z)} \le e^{Cd_g(y,z)}.$$

Next we get, letting $\hat{A} := A/|A| \in \mathcal{W}$ and $n_A := [|A|]$ be the integral part of $|A|$, that there are constants $C_0, \dots, C_3 > 0$ such that for all with $A \in \mathcal{C}_T$, $j \le k_\epsilon(A)$, $\tilde{y}, \tilde{z}$ on $W^s_{\mathrm{loc}}(y_j(A))$

$$\begin{aligned}\log\frac{|\det(d\varphi_1^A|_{E_s})(\tilde{y})|}{|\det(d\varphi_1^A|_{E_s})(\tilde{z})|} &\le C_0\sum_{n=0}^{n_A}\Big|\log|\det(d\varphi_1^{\hat{A}}|_{E_s})(\varphi_n^{\hat{A}}(\tilde{y}))| - \log|\det(d\varphi_1^{\hat{A}}|_{E_s})(\varphi_n^{\hat{A}}(\tilde{z}))|\Big| \\ &\le C_1\sum_{n=0}^{n_A}d_g(\varphi_n^{\hat{A}}(\tilde{y}), \varphi_n^{\hat{A}}(\tilde{z})) \\ &\le C_2\sum_{n=0}^{n_A}e^{-c_0 n}d_g(\tilde{y},\tilde{z}) \le C_3 d_g(\tilde{y},\tilde{z})\end{aligned}$$

where $c_0 > 0$ is less than the minimal contraction exponent of the flow in the direction $\hat{A} \in \mathcal{C}$ in the stable bundle. Such a uniform constant exists because the cone $\mathcal{C}$ does not touch the boundary of the Weyl chamber. Applying this with $\tilde{y} = H^G_{x_1,y_j}(y)$ and $\tilde{z} = H^G_{x_1,y_j(A)}(z)$ we obtain the desired estimate (1.28) since the holonomy map has uniformly Lipschitz bound wrt $A \in \mathcal{C}_T$ and $j \le k_\epsilon(A)$.

Now, this clearly implies that there is $C>0$ such that for all $y,z\in W^s_{\rm loc}(x_1)\cap V$

$$\frac{\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A(z)\,dA}{\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A(y)\,dA}\leq e^{Cd_g(y,z)}.$$

Coming back to (1.27) we get that

$$\frac{\nu^T(\mathcal{V}_\epsilon)}{\nu^T(F_\epsilon)}=\frac{\int_V\Big(\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A\,dA\Big)dL^s_{x_1}}{\int_{W^s_{\rm loc}(x_1)}\Big(\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A\,dA\Big)dL^s_{x_1}}$$

and the density on $W^s_{\rm loc}(x_1)$

$$d_\epsilon(y)=\frac{\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A(y)\,dA}{\int_{W^s_{\rm loc}(x_1)}\frac{1}{|\mathcal{C}_T|}\int_{\mathcal{C}_T}\sum_{j=1}^{k_\epsilon(A)}\rho_j^A(y)\,dA\,dL^s_{x_1}}$$

is of mass 1 for $L^s_{x_1}$ and its logarithm has a Lipschitz constant which is bounded independently of $T$ and $\epsilon$, so it satisfies $\alpha\leq d_\epsilon\leq\beta$ for some positive constant $\alpha,\beta$ uniform in $\epsilon,T$. This implies that

$$\alpha L^s_{x_1}(V)\leq\frac{\nu^T(\mathcal{V}_\epsilon)}{\nu^T(F_\epsilon)}=\int_V d_\epsilon(y)dL^s_{x_1}(y)\leq\beta L^s_{x_1}(V)$$

which shows, by letting $T\to\infty$ and then $\epsilon\to 0$, that the conditional measure $\nu^\infty_{x_1}$ is absolutely continuous with respect to Lebesgue $L^s_{x_1}$ on $W^s_{\rm loc}(x_1)$.

Next, by Ledrappier-Young [LY85, p. 533], if $\rho(y)$ is the density of the conditional $\nu^\infty_{x_1}$ with respect to Lebesgue $L^s_{x_1}$, one has for $A_0\in\mathcal{W}$ and $x,y\in W^s_{\rm loc}(x_1)$

$$\frac{\rho(y)}{\rho(x)}=\frac{\prod_{j=1}^\infty|\det(d\varphi_{-1}^{A_0}|_{E_s})|(\varphi_j^{A_0}(y))}{\prod_{j=1}^\infty|\det(d\varphi_{-1}^{A_0}|_{E_s})|(\varphi_j^{A_0}(x))}$$

Then, using this formula, an argument using the contraction on stable leaves and the chain rule for derivatives very similar to the proof of Proposition 1.7 shows that $\rho$ must be smooth on $W^s_{\rm loc}(x_1)$. A detailed proof is done in [dlL92, Corollary 4.4 and Lemma 5.5]. □

## 2. Equivalent characterisation of SRB measures

In this section we will study the measures $\mu_v$ obtained by the Ruelle-Taylor spectral theory. Let us first introduce some notation:

**Definition 2.1.** For an invariant measure $\mu$ define the *basin* to be those points $x\in\mathcal{M}$ such that for any $f\in C^0(\mathcal{M})$ and any proper subcone $\mathcal{C}\subset\mathcal{W}$ we have

$$\mu(f)=\lim_{T\to\infty}\frac{1}{|\mathcal{C}_T|}\int_{A\in\mathcal{C}_T}f(\varphi_{-1}^A(x))dA. \tag{2.1}$$

We say that an invariant measure $\mu$ is an *SRB measure* for $\tau$ if the basin has positive Lebsegue measure.

We show that they are linear combinations of SRB measures and give various other different characterisations:

**Theorem 3.** *Let $\tau$ be a smooth, locally free, $\mathbb{R}^\kappa$ Anosov action with generating map $X$, then we have:*

1) *The linear span over $\mathbb{C}$ of the SRB measures can be identified with the finite dimensional space* $\ker d_X|_{C^{-\infty}_{E_s^*}}$.
2) *The physical measure* (0.1) *is a linear combination of SRB measures.*
3) *The union of the basins of the SRB measures has full Lebesgue measure in* $\mathcal{M}$.
4) *An ergodic Radon probability measure* $\mu$ *is a SRB measure if and only if it is invariant by* $\varphi_t^A$ *for all* $A \in \mathfrak{a}$ *and has wave-front set* $\mathrm{WF}(\mu) \subset E_s^*$.
5) *The conditional measures of the SRB measures of an Anosov abelian action are absolutely continuous with respect to the Lebesgue measure on the stable manifolds, and they have smooth densities with respect to Lebesgue. Vice versa any ergodic invariant Radon probability measure that is absolutely continuous with respect to the Lebesgue measure (without assuming smooth densities) is a SRB measure.*

We furthermore prove that if the Anosov action is transitive then there is a unique SRB-measure (see Corollary 2.4) and if it is positively transitive the SRB-measure has full support (see Proposition 2.10). Theorem 3 together with these two additional result then gives Theorem 1 stated in the introduction.

We will first study the ergodic decomposition of $\mu_1$ and identify the basins of attractions (recall that the notion of basin of attraction was defined in the Introduction). We will use some arguments in the spirit of [BKL02, Proposition 2.3.2] to obtain

**Lemma 2.2.** *Let* $X$ *be an Anosov action, and let* $\mu = \mu_1$ *be the physical measure. There exist disjoint measurable sets* $F_1, \dots, F_r$ *such that* $\mu(F_i) = v_g(F_i) \neq 0$ *for all* $i$ *and* $\mu(\cup_i F_i) = v_g(\cup_i F_i) = 1$. *Furthermore the ergodic components of* $\mu$ *are given by* $\mu^i := \frac{\mathbf{1}_{F_i}}{\mu(F_i)} \cdot \mu$ *and* $F_i$ *is the basin of* $\mu_i$. *In particular each* $\mu_i$ *is an SRB measure. Finally the* $\mu_i$ *form a basis of the space* $\Pi^*(C^\infty(\mathcal{M}))$.

*Proof.* For $\mathcal{C} \subset \mathcal{W}$ a proper open subcone, $e_1 \in \mathfrak{a}^*$ with $e_1 > 0$ in $\mathcal{C}$, and $f \in C^0(\mathcal{M})$, we define

$$\Omega(f, \mathcal{C}) := \left\{ x \in \mathcal{M} \;\middle|\; f_-(x) := \lim_{T \to +\infty} \frac{1}{|\mathcal{C}_T|} \int_{A \in \mathcal{C}_T} f(\varphi_{-1}^A(x)) dA \text{ exists in } \mathbb{R} \right\}.$$

where $\mathcal{C}_T = \mathcal{C} \cap \{e_1(A) \in (0, T)\}$. It follows from the ergodic Birkhoff Theorem for actions (see [Bew71, Theorem 3]) that for all such $\mathcal{C}$ and $f$, and every invariant Borel measure $\nu$, $\Omega(f, \mathcal{C})$ has full $\nu$-measure. By a dichotomy argument on the cones, and using the fact that $C^0(\mathcal{M})$ is separable, we can improve this, by saying that

$$\Omega := \bigcap_{f, \mathcal{C}} \Omega(f, \mathcal{C}),$$

also has full $\nu$ measure for every invariant Borel measure $\nu$. Additionally, we observe that if $x \in \Omega$, then the weak unstable manifold satisfies $W^{wu}(x) \subset \Omega$.

More generally, if a set $F$ is a union of full weak unstable manifolds, we will say that $F$ is *unstable-invariant*. Notice that $\Omega$ is unstable-invariant. According to Lemma 1.9, this implies $\mathrm{WF}(\mathbf{1}_F) \subset E_u^*$, and thus $\mathbf{1}_F \in \mathcal{H}^{NG}$ for $N$ large enough. In particular, since $X_A \mathbf{1}_F = 0$ for all $A \in \mathfrak{a}$, $\mathbf{1}_F$ belongs to the finite dimensional space $\operatorname{ran} \Pi$ by (1.23). This means that $\mathbf{1}_F = \Pi \mathbf{1}_F$ in the distribution sense, thus *Lebesgue* almost everywhere since $\mathbf{1}_F$ is $L^1$.

Since for $v \in C^\infty$, the identity $\mu_v(u) = \langle \Pi u, v\rangle$ extends from smooth functions $u$ to elements $u \in \mathcal{H}^{NG}$, we have for each unstable-invariant set $F$,

$$\mu_v(F) = \int_{\mathcal{M}} (\Pi \mathbf{1}_F) v \, dv_g = \int_F v \, dv_g. \tag{2.2}$$

In particular, since $\mu(\Omega) = 1$, $\Omega$ has full Lebesgue measure (even if the Lebesgue measure is not invariant by the action). If $F$ is unstable invariant, then we also have $\mu(F) = v_g(F)$. Since each such $\mathbf{1}_F$ is an element of the space of resonant states at $\lambda = 0$

$$H := \mathrm{Span}\,\{\mathbf{1}_F \mid F \text{ is unstable invariant }\} / \sim \text{ is a subspace of } \operatorname{ran}\Pi.$$

(here, $\sim$ is the equivalence relation of being equal Lebesgue a.e. or, equivalently $\mu$ a.e.). Accordingly, we can find pairwise disjoint unstable invariant sets $\tilde{F}_1, \dots, \tilde{F}_r$ with $r \le \operatorname{rank}\Pi$, so that the $[\mathbf{1}_{\tilde{F}_j}]$ form a basis of $H$. Since $\Omega$ has full-measure, we can assume that $\cup \tilde{F}_j = \Omega$.

If $f$ is a continuous function and $\mathcal{C} \subset \mathcal{W}$ is an open proper subcone, we observe that the sets $\{x \in \Omega \mid f_-(x) \le r\}$ are unstable-invariant for each $r \in \mathbb{R}$, so they are finite unions of $\tilde{F}_j$'s, up to Lebesgue-null sets and by (2.2) also up to $\mu$-null sets. This implies that $f_-(x)$ is constant on each $\tilde{F}_j$, $\mu$-a.e. and we denote $f_{-,j}$ that value. By Lebesgue theorem and the invariance of $\mu$ by $\varphi^A_{-1}$,

$$\mu(\tilde{F}_j) f_{-,j} = \int_{\tilde{F}_j} f_-(x) d\mu(x) = \lim_{T\to\infty} \int_{\tilde{F}_j} \frac{1}{|\mathcal{C}_T|} \int_{\mathcal{C}_T} f(\varphi^A_{-1}(x)) dA \, d\mu(x) = \int_{\tilde{F}_j} f \, d\mu. \tag{2.3}$$

Thus, if we define $\mu^j := \mathbf{1}_{\tilde{F}_j} \mu / \mu(\tilde{F}_j)$ we get $f_{-,j} = \mu^j(f)$. Thus for arbitrary $f \in C^0(\mathcal{M})$ and an arbitrary proper subcone $\mathcal{C} \subset \mathcal{W}$ we have shown that for $\mu$ a.e. $x \in \tilde{F}_j$ we have $f_-(x) = \mu_j(f)$. Using as above that $C^0(\mathcal{M})$ is separable and that we can approximate an arbitrary cone by a countable number of cones, we deduce that the basin $F_j$ of $\mu_j$ differs from $\tilde{F}_j$ by a $\mu$ nullset or equivalently a Lebesgue nullset.

The same argument as in (2.3) can be done for $\mu_v$, so we deduce that for $v \in C^\infty$,

$$\int f d\mu_v = \sum_j f_-(F_j)\mu_v(F_j) = \sum_j \mu^j(f) \int_{F_j} v dv_g. \tag{2.4}$$

We have thus seen that the $\mathbf{1}_{F_j}$ form a meaningful basis of $H$. Now we prove that in fact $H = \operatorname{ran}\Pi$. Let $\pi : L^1(\mathcal{M}, \mu) \to L^1(\mathcal{M}, \mu)$ be the projector onto the set of $X_A$ invariant functions (for all $A \in \mathfrak{a}$) along the closed subspace generated by coboundaries $\{\varphi^A_1 f - f \mid f \in L^1(\mu),\ A \in \mathfrak{a}\}$. By the ergodic theorem of [Bew71], $\pi$ is a continuous operator and for $\mu$ almost all $x \in \mathcal{M}$

$$\pi f(x) = \lim_{T\to\infty} \frac{1}{|\mathcal{C}_T|} \int_{\mathcal{C}_T} f(\varphi^A_{-1}(x)) dA.$$

We have just proved that $\pi$ maps $C^0(\mathcal{M})$ to functions constant on the $F_j$'s. In particular, by density of continuous functions in $L^1(\mu)$, we deduce that the image of $\pi$ only contains functions constant on the $F_j$'s. This proves that the $F_j$, or more precisely, the $\mu^j$ are the ergodic components of $\mu$, and that Equation (2.4) is the ergodic decomposition (in the sense of Theorem 4.2.6 of [HK02]). One consequence of the above is that $\mu$ has at most $\operatorname{rank}\Pi$ ergodic components. However, in $\{\mu_v \,|\, v \in C^\infty(\mathcal{M})\}$, we can find $\operatorname{rank}\Pi$ linearly independent probability measures, absolutely continuous with respect to $\mu$, and invariant under the action. This implies that the number of ergodic components is at least $\operatorname{rank}\Pi$. We deduce that $H = \operatorname{ran}\Pi$, and that the $\mathbf{1}_{F_j}$ form a basis of $\operatorname{ran}\Pi$.

It remains to show that the $\mu_j$'s form a basis of $\Pi^*(C^\infty)$. Since they have pairwise disjoint support, the $\mu_j$ are linearly independent and they span a space of the same dimension as $\Pi^*(C^\infty)$. It thus remains to prove that all $\mu_j$ lie in $\Pi^*$ which by Proposition 1.12 can be achieved by considering the wavefront sets of the $\mu^j$: We can refine (2.4), since the LHS of (2.4) is equal to $\langle \Pi f, v\rangle$. We obtain

$$\Pi = \sum_{j=1}^{r} \mathbf{1}_{F_j} \otimes \omega_j^*, \tag{2.5}$$

for some $\omega_j^* \in (\mathcal{H}^{NG})^*$, $j = 1, \ldots, \operatorname{rank}\Pi$. Now, we can write $\mu_v$ in two different ways:

$$\mu_v(u) = \sum_j w_j^*(u) \int_{F_j} v dv_g = \sum_j \mu^j(u) \int_{F_j} v dv_g,$$

so that $w_j^* = \mu^j$. By Lemma 1.11, this shows that $\mu^j$ has wavefront set only in $E_s^*$, while we had a priori no information regarding the wavefront set of $\mu^j$. □

We thus have shown that the basins $F_j$ of $\mu^j$ cover $\mathcal{M}$ up to a Lebesgue zero set. Accordingly any SRB-measure must be equal to one of the $\mu^j$ and thus lie in $\{\mu_v \mid v \in C^\infty(\mathcal{M}),\ v \geq 0\}$. In order to prove the uniqueness of SRB-measures for transitive actions, we will prove.

**Lemma 2.3.** *For any SRB-measure $\mu_j$ with basin $F_j$ there is an open set $U_j \subset \mathcal{M}$ such that $\nu_g(U_j \cap F_j) = \nu_g(U_j)$.*

**Corollary 2.4.** *It the Anosov action $\tau$ transitive then there is a unique SRB measure.*

*Proof.* Assume that there are two SRB-measures $\mu_1$ and $\mu_2$ and $U_1, U_2$ the two open sets. By the transitivity there is $A \in \mathfrak{a}$ such that $U_1 \cap \varphi_1^A(U_2) \neq \emptyset$. Then we deduce that Lebesgue a.e. $x \in U_1 \cap \varphi_1^A(U_2)$ lies in $F_1$ and (by absolute continuity of $\phi_1^A$ w.r.t. Lebesgue) Lebesgue a.e. $x \in U_1 \cap \varphi_1^A(U_2)$ lies in $\varphi_1^A(F_2)$. But as the basins are flow invariant this is not possible except if $F_1 = F_2$. □

In order to prove Lemma 2.3 we first show:

**Lemma 2.5.** *For $x_0 \in \mathcal{M}$ let $L_{x_0}^s$ be the Lebesgue measure on $W_{\mathrm{loc}}^s(x_0)$ and call $\mathcal{L}$ the set of all these measures supported on small pieces of stable manifolds. Then the vector space $\Pi^*(C^\infty(\mathcal{M})) = \operatorname{span}\{\mu^1, \ldots, \mu^j\}$ is also equal to*

$$\operatorname{span}\Big\{\mu_{fL_{x_0}^s} \,\Big|\, x_0 \in \mathcal{M},\ f \in C_c^\infty(W_{\mathrm{loc}}^s(x_0), \mathbb{R}^+),\ \int_{W_{\mathrm{loc}}^s(x_0)} f dL_{x_0}^s = 1, L_{x_0}^s \in \mathcal{L}\Big\}.$$

*Proof.* We already know that all $\mu_{fL_{x_0}^s}$ are contained in the finite dimensional complex vector space $\Pi^*(C^\infty(\mathcal{M})) = \operatorname{span}\{\mu^1, \ldots, \mu^j\}$. Suppose that they do not span the whole space, then all $\mu_{fL_{x_0}^s}$ have to lie in a proper subspace and there is a linear functional on $\operatorname{span}\{\mu^1, \ldots, \mu^j\}$ vanishing on all $\mu_{fL_{x_0}^s}$. This would imply that there exists $w = \sum_j \lambda_j \mathbf{1}_{F_j}$, with $\lambda_j \in \mathbb{C}$ and $F_j$ defined by (1.22), such that $\langle w, fL_{x_0}^s\rangle = 0$ for any $L_{x_0}^s$ and $f$ as above. First take $v \in C^\infty(\mathcal{M}, \mathbb{R}^+)$ so that $\langle w, v\rangle \neq 0$. We can decompose $v = \sum_i \chi_i v$ in small charts $(U_i)_i$ where the disintegration along stable leaves can be performed as in (1.10): taking $G_i$ some local transverse manifolds to the $W^s$ foliation in $U_i$,

$$\begin{aligned}\langle w, \chi_i v\rangle &= \int_{U_i} \chi_i(x)v(x)w(x)\, dv_g(x) \\ &= \int_{G_i}\int_{W^s_{\mathrm{loc}}(z)} \chi_i(y)w(y)v(y)\delta_{W^s_{\mathrm{loc}}(z)}(y)dL^s_z(y)\, dL^{G_i}(z) \\ &= 0\end{aligned}$$

where we used our assumption with $f = \chi_i v\delta_{W^s_{\mathrm{loc}}(z)}$ on $W^s_{\mathrm{loc}}(z)$ and $x_0 = z$ to get the last line. We obtain a contradiction. □

We deduce, using Proposition 1.14 the

**Corollary 2.6.** *The conditional measures $\mu^j$ are absolutely continuous with respect to the Lebesgue measure on the stable manifolds, and they have smooth densities with respect to Lebesgue.*

We can now prove the Lemma 2.3.

*Proof of Lemma 2.3.* Recall that $\mu^j := \frac{\mathbf{1}_{F_j}}{\mu(F_j)}\mu$ and that $F_j$ are the basins of $\mu^j$. Let us construct these open sets $U_j$: We consider a point $x_0 \in \operatorname{supp}\mu^j$ and a small product neighborhood $V$ of $x_0$. Furthermore consider the disintegration of $\mu^j$ along the strong stable foliation in $V$. By Corollary 2.6, this gives locally around $x_0$ for any $y \in W^{wu}_{\mathrm{loc}}(x_0)$ a measure $\mu^j_y$ on $W^s_{\mathrm{loc}}(y)$ with a smooth density with respect to the Lebesgue measure $L^s_y$ on $W^s_{\mathrm{loc}}(y)$ and a transversal measure $\hat{\mu}^j$ on $W^{wu}_{\mathrm{loc}}(x_0)$. But we know that $\mathbf{1}_{F_j}\mu^j = \mu^j$ and consequently for $\hat{\mu}^j$ a.e. $y \in W^{wu}_{\mathrm{loc}}(x_0)$ we have $\mathbf{1}_{F_j\cap W^s_{\mathrm{loc}}(y)}\mu^j_y = \mu^j_y$. In particular there is at least one $y_0 \in W^{wu}(x_0)$ such that $\mathbf{1}_{F_j\cap W^s_{\mathrm{loc}}(y_0)}\mu^j_{y_0} = \mu^j_{y_0} \neq 0$. But as $\mu^j_{y_0}$ has a smooth density with respect to the Lebesgue measure on $W^s_{\mathrm{loc}}(y_0)$ there exists an nonempty open set $U^j_{y_0} := \operatorname{int}(\operatorname{supp}\mu^j_{y_0}) \subset W^s_{\mathrm{loc}}(y_0)$ such that $\mu^j_{y_0}(F_j \cap W^s_{\mathrm{loc}}(y_0)) = \mu^j_{y_0}(U^j_{y_0})$. As the $F_j$ are invariant in the weak-unstable directions we can consider the open set $U_j := \cup_{z\in U^j_{y_0}} W^{wu}_{\mathrm{loc}}(z) \subset V$. Using the absolute continuity of the weak unstable foliation, one checks that $v_g(F_j \cap U_j) = v_g(U_j)$.

□

In a very similar way as the one presented above for the construction of the $U_j$ we prove

**Lemma 2.7.** *Let $\nu$ be an ergodic Radon measure on $\mathcal{M}$ that has an absolutely continuous disintegration w.r.t. $W^s_{\mathrm{loc}}$ then the basin of $\nu$ has positive Lebesgue measure.*

*Proof.* Let $\Omega$ be the set of $x \in \mathcal{M}$ such that for all $f \in C^0(\mathcal{M})$ and all cones $\mathcal{C} \subset \mathcal{W}$

$$f_-(x) = \lim_{T\to+\infty} \frac{1}{|\mathcal{C}_T|}\int_{A\in\mathcal{C}_T} f(\varphi^A_{-1}(x))dA = \nu(f)$$

By the ergodicity assumption $\Omega$ has full $\nu$ measure and is in particular a nonempty unstable invariant set. Moreover $\mathbf{1}_\Omega\nu = \nu$. We now consider a point $x_0 \in \Omega$ in the support of $\nu$ and a small neighborhood $V$ of $x_0$. Furthermore consider the disintegration of $\nu$ along the strong stable foliation in $V$: This gives locally around $x_0$ for any $y \in W^{wu}_{\mathrm{loc}}(x_0)$ a measure $\nu_y$ on $W^s_{\mathrm{loc}}(y)$ with a density with respect to the Lebesgue measure $L^s_y$ on $W^s_{\mathrm{loc}}(y)$ and a transversal measure $\hat{\nu}$ on $W^{wu}_{\mathrm{loc}}(x_0)$. But we know that $x_0$ is in the support of $\nu$ and consequently there is at least one $y_0 \in W^{wu}_{\mathrm{loc}}(x_0)$ such that $\mathbf{1}_{\Omega\cap W^s_{\mathrm{loc}}(y_0)}\nu_{y_0} = \nu_{y_0} \neq 0$. As $\nu_{y_0}$ is absolutely continuous w.r.t $L^s_{y_0}$ we conclude that $\Omega \cap W^s_{\mathrm{loc}}(y_0)$ has positive $L^s_{y_0}$ measure. As the $\Omega$ is invariant in the weak-unstable directions and as the holonomies along these weak unstable directions are absolutely continuous we deduce, that $\Omega \cap W^s_{\mathrm{loc}}(y)$ has positive $L^s_y$ measure for all $y$ in $V$. Now we can use that also the Lebesgue measure has absolutely continuous disintegration along $W^s_{\mathrm{loc}}$ and conclude that $V \cap \Omega$ has positive Lebesgue measure. □

We already know that for a topologically transitive Anosov action only the physical measure can have a basin of positive Lebesgue measure. Consequently for a topologically transitive Anosov action any ergodic measure with absolutely continuous disintegration w.r.t. $W^s_{\mathrm{loc}}$ is automatically the physical measure. For the general case we consider:

**Lemma 2.8.** *Let $\nu$ be a Radon measure on $\mathcal{M}$ that has an absolutely continuous disintegration w.r.t. $W^s_{\mathrm{loc}}$ and is invariant under the Anosov action $\tau$. If $\mathrm{supp}(\nu) \subset \mathcal{M}$ is connected, then $\nu$ is ergodic, i.e. there is a full $\nu$ measure set $F \subset \mathcal{M}$ such that for all $x \in F$, all $f \in C^0(\mathcal{M})$ and all proper subcones $\mathcal{C} \subset \mathcal{W}$ one has $f_-(x) = \int_{\mathcal{M}} f d\nu$.*

*Proof.* Let $\Omega$ be the set of $x \in \mathcal{M}$ such that for all $f \in C^0(\mathcal{M})$ and all cones $\mathcal{C} \subset \mathcal{W}$

$$f_-(x) = \lim_{T\to+\infty} \frac{1}{|\mathcal{C}_T|}\int_{A\in\mathcal{C}_T} f(\varphi^A_{-1}(x))dA = \nu(f)$$

Then we know by the reasoning at the beginning of Lemma 2.2 that $\Omega$ has full $\nu$ measure. The function $f_-$ is thus a measurable function that is constant on weak unstable leaves. Let $A \in \mathcal{W}$, then $W^s(x)$ are precisely the stable sets of the diffeomorphism $\varphi^A_1$. By [BS02, Lemma 6.3.2] there is a $\nu$ nullset $N$ such that for any $x \in \mathcal{M}\setminus N$, $f_-$ is constant on $W^s(x)\setminus N$.

Now by the absolute continuity in any product neighbourhood $U \subset \mathcal{M}$ we can write

$$0 = \int_U \mathbf{1}_N d\nu = \int_{W^{wu}_{\mathrm{loc}}(m)}\left(\int_{W^s_{\mathrm{loc}}(y)} \mathbf{1}_N(z)\delta_{W^s}(z)dL^s_y(z)\right)d\hat{\nu}(y)$$

Accordingly for $\hat{\nu}$ almost all $y \in W^{wu}_{\mathrm{loc}}(m)$ one has $N \cap W^s_{\mathrm{loc}}(y)$ is a $L^s_y$ zero set. Pick one such $y_0$ and $z_0 \in W^s_{\mathrm{loc}}(y_0)$. Then, by the definition of $N$ for all $z \in W^s_{\mathrm{loc}}(y_0)\setminus N$ we have $f_-(z) = f_-(z_0)$. Let $y_1 \in W^{wu}_{\mathrm{loc}}(m)$ by any other point. As $f_-$ is constant along the weak unstable leaves we can use the holonomy along the weak unstable leaves $H^{W^{wu}}_{y_0,y_1} : W^s(y_0) \to W^s(y_1)$. As we know that this holonomy is absolutely continuous we have that

$$\{f_- = f_-(z_0)\} \cap W^s_{\mathrm{loc}}(y_1) \subset H^{W^{wu}}_{y_0,y_1}(W^s_{\mathrm{loc}}(y_0)) \subset W^s_{\mathrm{loc}}(y_1)$$

has full $L^s_{y_1}$ measure. Using once more the absolute continuity of $\nu$ along the stable foliation we get

$$\nu(\{f_- = f_-(z_0)\} \cap U) = \nu(U) \tag{2.6}$$

Using the connectedness of $\mathrm{supp}(\nu)$, we deduce that $F_f := \{f_- = f_-(z_0)\}$ has full $\nu$ measure and from [Bew71, Theorem 1 and 3] we get

$$f_-(z_0) = \int_{\mathcal{M}} f_- d\nu = \int f d\nu.$$

We finally take $F = \cap_{f\in C^0(\mathcal{M})} F_f$ and use he fact that $C^0(\mathcal{M})$ is separable to reduce the intersection to a countable set in $C^0(\mathcal{M})$, we obtain that $F$ has full $\nu$ measure and satisfies the desired property. □

Consequently for any Radon $\operatorname{supp}\nu$ measure on $\mathcal{M}$ that has an absolutely continuous disintegration w.r.t. $W^s_{\mathrm{loc}}$ and is invariant under the Anosov action $\tau$ we can decompose the support $\operatorname{supp}\nu$ into its connected compontents $C_i$ and we deduce that $\nu_i := \frac{\mathbf{1}_{C_i}}{\nu(C_i)}\nu$ are SRB-measures. This proves property 5) of Theorem 3.

**Definition 2.9.** We call an Anosov action *positively transitive* if there is a proper subcone $\mathcal{C} \subset \mathcal{W}$ such that for any two open sets $U, V \in \mathcal{M}$ there is $A \in \mathcal{C}$ such that $\varphi_1^A(U) \cap V \neq \emptyset$.

If the Anosov action is positively transitive then it is obviously transitive and we know that there is a unique SRB measure.

**Proposition 2.10.** *If the Anosov action is positively transitive then the SRB measure $\mu$ has full support.*

*Proof.* Assume that $\operatorname{supp}\mu \neq \mathcal{M}$ then there are $m_0 \in \mathcal{M}$, $\varepsilon, \delta > 0$ s.t. the product neighbourhood (cf. (1.8)) $U_0 := B_{W^s}(m_0, \delta) \times B_{W^{wu}}(m_0, \varepsilon)$ is disjoint from $\operatorname{supp}\mu$. Let us also define the set $U_1 := B_{W^s}(m_0, \delta) \times B_{W^{wu}}(m_0, \varepsilon/2)$. Recall that, after chosing an arbitrary norm on $\mathfrak{a}$, the transversal hyperbolicity of the Anosov action implies that there are $C', \nu > 0$ such that for all $A \in \mathcal{C}, v \in E_u$ we have $\|d\varphi_1^A v\|_g > (1/C')e^{\nu|A|}\|v\|_g$. As the splitting $E_0 \oplus E_s \oplus E_u$ is continuous and $\mathcal{M}$ is compact, there is $C > 1$ so that $g(v_0, v_u) \leq (1 - 1/C)\|v_0\|_g\|v_u\|_g$ for all $v_u \in E_u, v_0 \in E_0$, and we deduce that there is $C > 1$ for all $A \in \mathcal{C}$ and $v \in E_0 \oplus E_u$ we have $\|d\varphi_1^A v\|_g \geq (1/C)\|v\|_g$. As a direct consequence we deduce that for all $A \in \mathcal{C}$ and $\tilde{m} \in \varphi_1^A(U_1)$ there is $\tilde{\delta} > 0$ such that

$$B_{W^s}(\tilde{m}, \tilde{\delta}) \times B_{W^{wu}}(\tilde{m}, \varepsilon/(2C)) \subset \varphi_1^A(U_0)\,. \tag{2.7}$$

Since the stable and weak unstable foliations are continuous, taking $\eta > 0$ small enough, we can ensure that for $m \in \mathcal{M}$ and $y \in B_{W^s}(m, \eta)$, the weak unstable ball $B_{W^{wu}}(y, \varepsilon/(2C))$ is large enough, in the sense that

$$B_{W^{wu}}(y, \varepsilon/(2C)) \ \cap\ B_{W^s}(m, \eta) \times B_{W^{wu}}(m, \varepsilon/(4C)) \ \subset\ B_{W^{wu}}(y, \varepsilon/(2C)).$$

Let us thus cover the manifold $\mathcal{M}$ with a finite open cover of product neighbourhoods $V_j = B_{W^s}(m_j, \eta) \times B_{W^{wu}}(m_j, \varepsilon/(4C))$. As the SRB measure $\mu$ is nonzero, there is at least one $V_k$ such that $\mu|_{V_k} \neq 0$. By the positive transitivity of the Anosov action we can find $A \in \mathcal{C}$ and $\tilde{m} \in \varphi_1^A(U_1) \cap V_k \neq \emptyset$. By (2.7) applied with $\tilde{m}$ and since $B_{W^{wu}}(\tilde{m}, \varepsilon/(2C))$ is large enough, there is an open set $O = B_{W^s}(\tilde{m}, \tilde{\delta}) \cap B_{W^s}(m_k, \eta)$ such that

$$O \times B_{W^{wu}}(m_k, \varepsilon/(4C)) \subset V_k \cap \varphi_1^A(U_0).$$

Recall that we started with the assumption that $\mu(U_0) = 0$ and we deduce by the invariance under the Anosov action that $\mu(O \times B_{W^{wu}}(m_k, \varepsilon/(4C))) = 0$. But the latter cannot be true, because by the absolute continuity of the SRB measure (Proposition 1.14) we can desintegrate $\mu$ in the product neighbourhood $V_k$ and obtain

$$\mu(O \times B_{W^{wu}}(m_k, \varepsilon/(4C))) = \int_{B_{W^{wu}}(m_k, \varepsilon/(4C))} \left( \int_{W^s_{\mathrm{loc}}(y)} \mathbf{1}_{H^{W^{wu}}_{m_k,y}(O)}(z)\rho_y(z)dL^s_y(z) \right) d\hat{\mu}(y)$$

for some smooth positive $\rho$. Now $\mu(O \times B_{W^{wu}}(m_k, \varepsilon/(4C))) = 0$ would imply that

$$\int_{W^s_{\mathrm{loc}}(y)} \mathbf{1}_{H^{W^{wu}}_{m_k,y}(O)}(z)\rho_y(z)dL^s_y(z) = 0$$

for $\hat{\mu}$ almost all $y$, but this contradicts the fact that the conditional densities $\rho_y$ are strictly positive and that $H^{Wwu}_{m_k,y}(O)$ are nonempty open sets. □

## 3. A Bowen type formula for the SRB measure and Guillemin trace formula

In this section we show that the SRB measure $\mu$ can be expressed in terms of the periodic orbits of the flow. We obtain thus a generalized Bowen formula. Before we can state our result, we have to recall some basic facts regarding the structure of periodic orbits of Anosov actions. We recall the classical Lemma

**Lemma 3.1.** *Let $\tau$ be an Anosov action with positive Weyl chamber $\mathcal{W}$. Let $x \in \mathcal{M}$, and $A \in \mathcal{W}$, such that $\varphi_1^A(x) = x$. Then there exists a lattice $L \subset \mathbb{A}$, such that for all $A' \in L$, $\varphi_1^{A'}(x) = x$, and $T := \tau(\mathbb{A})x \simeq \mathbb{A}/L$ is an embedded torus in $\mathcal{M}$. We denote $L = L(T)$.*

*Proof.* It suffices to prove that the orbit of $x$ under the action is closed. We denote $Y$ this orbit. Since the action is abelian, $Y$ is comprised only of fixed points of $\varphi_1^A$, and thus so is $\overline{Y}$. However, since $A$ is transversely hyperbolic, we can deduce that for each fixed point $y$ of $\varphi_1^A$ there is an open set $U \ni y$ such that for $y' \in U$, if $\varphi_1^A(y') = y'$, then $y'$ is in the local orbit of $y$ under the action. This proves that $\overline{Y} = Y$. □

We stress that there may be periodic orbits $\{\varphi_t^{A_0}(x), t \in [0,1]\}$ of the flow in direction $A_0 \in \mathfrak{a}$ which are not contained in an invariant torus. However this can only happen if the orbit is periodic with respect to a direction $A_0$ which is *not* transversely hyperbolic and thus in no positive Weyl chamber. In any case, we denote by $\mathcal{T}$ the set of invariant tori which are precisely the compact orbits of the Anosov $\mathbb{A}$ action. According to the closing lemma (see [KS95, Theorem 2.4]), the periodic tori are locally discrete in the sense that for each compact set $K \subset \mathcal{W}$,

$$K \bigcap \Big( \bigcup_{T \in \mathcal{T}} L(T) \Big) \text{ is finite.} \tag{3.1}$$

Pushing forward the Lebesgue measure on $\mathfrak{a}$ by the action, we obtain a natural measure on orbits, and in particular on the periodic tori. It thus makes sense to talk of averages over periodic orbits.

Given $T \in \mathcal{T}$, $x \in T$, and $A \in L(T) \cap \mathcal{W}$, the map $\varphi_1^A$ is hyperbolic transversal to $T$ by definition of being an Anosov action. In particular, if we set $\mathcal{P}_A(x) := d_x(\varphi_{-1}^A)_{|E_u(x) \oplus E_s(x)}$, we find that

$$|\det(1 - \mathcal{P}_A(x))| \neq 0$$

and that it does not depend on $x$. We denote then by $\mathcal{P}_A$ some choice of $\mathcal{P}_A(x)$. Equipped with these notations, we can now state our result

**Theorem 4.** *Let $\tau$ be a transitive Anosov action, with Weyl chamber $\mathcal{W}$. Let $A_1 \in \mathcal{W}$ and $e_1 \in \mathfrak{a}^*$ such that $e_1(A_1) = 1$. Let $\mathcal{C} \subset \mathcal{W}$ be a small open cone containing $A_1$ and define $\mathcal{C}_{a,b} := \{A \in \mathcal{C} \,|\, e_1(A) \in [a,b]\}$ if $a, b > 0$. Let $\mu$ be the SRB measure and $a, b > 0$. Then for each $f \in C^\infty(\mathcal{M})$, we have*

$$\mu(f) = \lim_{N \to \infty} \frac{1}{|\mathcal{C}_{aN,bN}|} \sum_{T \in \mathcal{T}} \sum_{A \in \mathcal{C}_{aN,bN} \cap L(T)} \frac{\int_T f}{|\det(1 - \mathcal{P}_A)|}.$$

In the rank 1 situation, one way to prove such a formula is to consider the flat trace of the resolvent $(X - s)^{-1}$, relating it with the periodic orbits on the one hand via the Guillemin trace formula, and with the SRB measure on the other hand, using the spectral theory of $X$ on some anisotropic space.

In our case, the proof will be heuristically similar. However, it will be complicated by the fact that we do not have a resolvent at our disposal in the multiflow situation. Thankfully, we can work around this. In the paper [GBGHW20], we introduced some averaged propagators (see (1.20) for the case $\lambda = 0$)

$$R(\lambda) := \prod_{j=1}^{\kappa} \int_{\mathbb{R}^\kappa} e^{-t_j(X_j+\lambda_j)}\psi_j(t_j)dt.$$

where $X_j = X_{A_j}$ for some basis $(A_j)_j \in \mathcal{W}$ of $\mathfrak{a}$. By definition, $R(\lambda)$ commutes with the action. We proved that given $N > 0$, for $G$ well chosen, $R(\lambda)$ is quasi-compact on $\mathcal{H}^{NG}$ for all $\lambda$'s with $\operatorname{Re}\lambda_j > -N$, $j = 1, \ldots, \kappa$. Then we proved that given $\lambda_0$, if $\lambda$ is close enough to $\lambda_0$, $\lambda$ is a Ruelle-Taylor resonance of $X$ if and only if $\lambda$ is in the joint spectrum of the family $(X_1, \ldots, X_\kappa)$ acting on $\ker(R(\lambda_0) - 1)$ (see [GBGHW20, Proposition 4.17]). For this reason, the study of the Ruelle-Taylor resonances (and in particular 0, the leading resonance) can be done using the averaged propagators $R(\lambda)$. More generally, we will take functions $\psi \in C_c^\infty(\mathcal{W})$, with $\int \psi = 1$, and consider

$$R_\psi(\lambda) := \int_{\mathcal{W}} e^{-X_A - \lambda(A)}\psi(A)dA.$$

What will replace the propagator in our arguments will thus be the so-called *shifted resolvent* $T^\lambda_{\psi,f}(s)$, defined for $f \in C^\infty(M)$,

$$T^\lambda_{\psi,f}(s) := fR_\psi(\lambda)(R_\psi(\lambda) - s)^{-1}.$$

We will show that $T^\lambda_{\psi,f}(s)$ admits a flat trace, and express this flat trace in terms of the orbits. Practically, one would rather consider the resolvent $(R_\psi(\lambda) - s)^{-1}$ of $R_\psi(\lambda)$ than the shifted resolvent, but this operator would not satisfy the right wave-front set condition to define the flat trace.

3.1. **Guillemin trace formula.** To start with, we need to extend the Guillemin trace formula to the case of Anosov actions. We write $n := \dim \mathcal{M}$ and $\kappa$ the rank of the action $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$. We will follow the proof in rank 1 by Dyatlov-Zworski [DZ16]. Recall that the flat trace is a regularized trace for certain operators that are not trace class. The conormal to the diagonal $\Delta$ of $\mathcal{M} \times \mathcal{M}$ is given by

$$N^*\Delta = \{(x, x, \xi, -\xi) \mid x \in \mathcal{M},\ \xi \in T^*_x\mathcal{M}\} \subset T^*(\mathcal{M} \times \mathcal{M}).$$

If $P : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ is a continuous linear operator, one can consider its Schwartz kernel $\mathcal{K}_P \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$, and assuming that $\mathrm{WF}(\mathcal{K}_P) \cap N^*\Delta = \emptyset$ we can set

$$\mathrm{Tr}^\flat(P) := \langle \iota_\Delta^* \mathcal{K}_P, 1\rangle_{C^{-\infty}, C^\infty}$$

where $\iota_\Delta : x \in \mathcal{M} \mapsto (x, x) \in \Delta \subset \mathcal{M} \times \mathcal{M}$ is the inclusion. Here the pull-back is well-defined thanks to the wave-front condition, see [Hör03, Theorem 8.2.4].

**Proposition 3.2** (Guillemin Trace formula)**.** *Let $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ be an Anosov action with Weyl chamber $\mathcal{W}$. Then the map*

$$f \in C^\infty(\mathcal{M}), \psi \in C_c^\infty(\mathcal{W}) \mapsto \mathrm{Tr}^\flat\left(f \int_{\mathcal{W}} e^{-X_A}\psi(A)dA\right),$$

*is well defined, and extends as a Radon measure on $\mathcal{M}\times\mathcal{W}$. For each closed cone $\mathcal{C}\subset\mathcal{W}$, there exists a constant $C>0$ such that if $h\in C^0(\mathcal{M}\times\mathcal{W})$ satisfies* $\operatorname{supp} h\subset\mathcal{M}\times\mathcal{C}$ *and* $\sup_{x,\in\mathcal{M},A\in\mathcal{C}} e^{C|A|}|h(x,A)|<\infty$*, then*

$$\operatorname{Tr}^\flat\left(\int_{\mathcal{W}} e^{-X_A} h\, dA\right) = \sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)} \frac{\int_T h(x,A)dx}{|\det(1-\mathcal{P}_A)|}.$$

*Remark* 3.3. This formula has a direct extension to the action on vector bundles which we do not directly need in this article but nevertheless mention for completeness: instead of functions, let us consider the action on sections of some vector bundle $\mathcal{E}$. For $T\in\mathcal{T}$, and $A\in L(T)$, and $x\in T$, we denote by $M(A,x)$ the holonomy map on $\mathcal{E}_x$. Then for $f\in C^\infty(\mathcal{M}\times\mathcal{C})$ the formula becomes

$$\operatorname{Tr}^\flat_{\mathcal{E}}\left(\int_{\mathcal{W}} f e^{-X_A} dA\right) = \sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)} \frac{\int_T f(x,A)\operatorname{Tr} M(A,x)dx}{|\det(1-\mathcal{P}_A)|}.$$

Following the approach for flows in [GLP13, BWS20], this gives a method to get rid of the Poincaré factor: We define for $m\in[0,n-\kappa]$ the bundle

$$\mathcal{E}_0^m := \{\omega\in\Lambda^m T^*\mathcal{M}\,|\,\forall A\in\mathfrak{a},\ \iota_{X_A}\omega=0\}.$$

and denote by $o(E_s)$ the orientation bundle which is a flat line bundle (see e.g. [BWS20, Definition 1.4]). The Guillemin trace formula for the bundle $\mathcal{E}_0^m\otimes o(E_s)$ reads

$$\operatorname{Tr}^\flat\Big(\int f e^{-X_A}|_{\mathcal{E}_0^m\otimes o(E_s)}\, dA\Big) = \sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)} \frac{\operatorname{Tr}(\Lambda^m\mathcal{P}_A)\operatorname{sign}(\det(\mathcal{P}_A)_{|E_s})\int_T f(x,A)dx}{|\det(1-\mathcal{P}_A)|}.$$

Using

$$\det(1-\mathcal{P}_A) = \sum_{m=0}^{n-\kappa}(-1)^m\operatorname{Tr}(\Lambda^m\mathcal{P}_A) \text{ and } \operatorname{sign}(\det(1-\mathcal{P}_A)) = (-1)^{1+\dim E_s}\operatorname{sign}(\det{(P_A)}_{|E_s})$$

we get a new formula where the determinant of the Poincaré map disappears

$$\sum_{m=0}^{n-\kappa}(-1)^{m+\dim E_s}\operatorname{Tr}^\flat\Big(\int f e^{-X_A}|_{\mathcal{E}_0^m\otimes o(E_s)}\, dA\Big) = \sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)}\int_T f(x,A)dx. \tag{3.2}$$

*Proof of Proposition 3.2.* The proof is divided into three steps. The first step consists in checking the wavefront set condition necessary to define the flat trace. Next, we need to make a local explicit computation to obtain the formula. Finally, we need to obtain some estimates to extend the formula to non-compactly supported functions.

For the first two parts of the proof, we can assume to be working with $f(x)\psi(A)$, using the density of product functions in functions on $\mathcal{M}\times\mathcal{W}$. We introduce the notation

$$R_\psi := R_\psi(0) = \int_{\mathcal{W}} \psi(A) e^{-X_A} dA.$$

**First step:** Let us show that $R_\psi$ has a well defined flat trace, which means that its Schwartz kernel $\mathcal{K}_{R_\psi}$ satisfies

$$\operatorname{WF}(\mathcal{K}_{R_\psi})\cap N^*\Delta=\emptyset \tag{3.3}$$

if $\Delta \subset \mathcal{M}\times\mathcal{M}$ is the diagonal. First, we consider $e^{-X}$ as an operator $C_c^\infty(\mathcal{M})\otimes C_c^\infty(\mathcal{W})\to C^\infty(\mathcal{M})$ by $f\otimes\psi\mapsto R_\psi f$ and we consider its Schwartz kernel $\mathcal{K}_{e^{-X}}\in\mathcal{D}'(\mathcal{W}\times\mathcal{M}\times\mathcal{M})$. Using the formula for the wavefront set of a pushforward, we obtain

$$\mathrm{WF}(\mathcal{K}_{R_\psi})\subset\{(x,\eta,x',\eta')\in T^*(\mathcal{M}\times\mathcal{M})\,|\,\exists A\in\mathrm{supp}(\psi),(A,0,x,\eta,x',\eta')\in\mathrm{WF}(\mathcal{K}_{e^{-X}})\}.$$

Since $\mathcal{K}_{e^{-X}}(A,x,x')=\delta_{x=\varphi_1^A(x')}$, one has, by [Hör03, Theorem 8.2.4],

$$\mathrm{WF}(\mathcal{K}_{e^{-X}})\subset\big\{(A,-\eta(X_\bullet(\varphi_1^A(x'))),\varphi_1^A(x'),\eta,x',-\,d\varphi_1^A(x')^T\eta)\in T^*(\mathcal{W}\times\mathcal{M}\times\mathcal{M})\;\big|\;A\in\mathcal{W},x'\in\mathcal{M},\eta\in T^*_{\varphi_1^A(x')}\mathcal{M}\setminus\{0\}\big\}. \tag{3.4}$$

Thus a point of $\mathrm{WF}(\mathcal{K}_{R_\psi})$ belongs to $N^*\Delta$ if and only if there exists $x'\in\mathcal{M}$ and $A\in\mathrm{supp}\,\psi\subset\mathcal{W}$ such that $\varphi_1^A(x')=x'$, $\eta(X_{A'}(\varphi_1^A(x')))=0$ for all $A'\in\mathfrak{a}$ and $\eta=d\varphi_1^A(x')^T\eta\neq 0$. Note that $\eta(X_{A'}(\varphi_1^A(x')))=0$ implies that $\eta\in E_u^*\oplus E_s^*$, where $d\varphi_1^A(x')^T$ has no eigenvalues of modulus 1 by normal hyperbolicity. This shows that (3.3) holds.

For the **second step** we start with the following Lemma:

**Lemma 3.4.** *Let $x_0\in\mathcal{M}$ and $A_0\in\mathcal{W}$ such that $\varphi_1^{A_0}(x_0)=x_0$. There is a neighborhood $U$ of $x_0$ and $\epsilon>0$ such that if $B(A_0,\epsilon):=\{A\in\mathfrak{a}\,|\,|A-A_0|<\epsilon\}$ one has $\varphi_1^A(x_0)\in U$ for all $|A|<\epsilon$ and for each $h\in C_c^\infty(B(A_0,\epsilon)\times U)$,*

$$\mathrm{Tr}^\flat\Big(\int h(A,x)e^{-X_A}\,dA\Big)=\frac{1}{|\det(1-\mathcal{P}_{A_0}(x_0))|}\int_{|A|<\epsilon}h(A_0,\varphi_1^A(x_0))dA.$$

*Proof.* We follow the argument in [DZ16, Lemma B.1]. Take an arbitrary basis $A_1,\dots,A_\kappa$ of $\mathfrak{a}$ and take $\phi:U\to B_{\mathbb{R}^n}(0,\epsilon)$ some diffeomorphism so that, if $y=\phi(x)$

$$\phi(x_0)=0,\quad\forall x\in U,\,d\phi(x)X_{A_i}=\partial_{y_i}$$
$$d\phi(x_0)(E_u(x_0)\oplus E_s(x_0))=\mathrm{span}\{\partial_{y_{\kappa+1}},\dots,\partial_{y_n}\}.$$

Let $F:B_{\mathbb{R}^{n-\kappa}}(0,\epsilon)\to B_{\mathbb{R}^\kappa}(0,\epsilon_1)$ and $G:B_{\mathbb{R}^{n-\kappa}}(0,\epsilon)\to B_{\mathbb{R}^{n-\kappa}}(0,\epsilon_1)$ so that

$$\phi\circ e^{-X_{A_0}}\circ\phi^{-1}(0,y'')=(F(y''),G(y'')),\quad y''\in\mathbb{R}^{n-\kappa},|y''|<\epsilon.$$

and $F(0)=0,G(0)=0$. For $A\in U$ and $(y',y'')\in B_{\mathbb{R}^n}(0,\epsilon)$ we thus have, identifying $\mathfrak{a}$ with $\mathbb{R}^\kappa$

$$\phi\circ e^{-X_A}\circ\phi^{-1}(y',y'')=(-A+A_0+y'+F(y''),G(y'')).$$

Then for $(z',z'')$ and $(y',y'')$ in $B_{\mathbb{R}^n}(0,\epsilon)$, we can write, using that $\mathcal{K}_{e^{-X}}(A,x,x')=\delta_{x=\varphi_1^A(x')}$,

$$\mathcal{K}_{e^{-X}}(A,\phi^{-1}(z',z''),\phi^{-1}(y',y''))=\delta(G(z'')-y'')\delta(y'+A-A_0-z'-F(z'')).$$

Taking the flat trace gives

$$\begin{aligned}\mathrm{Tr}^\flat\Big(\int he^{-X_A}\,dA\Big)&=\int\Big(h(A,\phi^{-1}(y',y''))\delta(y''-G(y''))\delta(A-A_0-F(y''))\Big)dy'dy''dA\\&=\int_{B(0,\epsilon)}h(A_0+F(y''),\phi^{-1}(y',y''))\delta(y''-G(y''))dy'dy''.\end{aligned}$$

Now $\mathrm{Id}-dG(0)$ is invertible by the normal hyperbolicity of the action (it is conjugated to the Poincaré map $d_{x_0}\varphi_{-1}^{A_0}|_{E_u\oplus E_s}$), thus $y''=G(y'')$ has a unique solution $y''=0$ for $|y''|<\epsilon$ if $\epsilon>0$ is small enough, and we thus get

$$\mathrm{Tr}^\flat\Big(\int he^{-X_A}\,dA\Big)=\frac{1}{|\det(1-d_{x_0}\varphi_{-1}^{A_0}|_{E_u\oplus E_s})|}\int_{|y'|<\epsilon}h(A_0,\phi^{-1}(y',0))dy'$$

this concludes the proof of Lemma 3.4. □

Now, call $T_{x_0}$ the periodic torus containing $x_0$, then if $h(A,x) = \psi(A)f(x) \in C_c^\infty(\mathcal{W}) \otimes C^\infty(\mathcal{M})$ is such that $f$ is supported in a small neighborhood of $T_{x_0}$ containing $x_0$, and

$$(\cup_{T\in\mathcal{T}\setminus T_{x_0}}(L(T)\times T)) \cap (\mathrm{supp}(\psi)\times\mathrm{supp}(f)) = \emptyset,$$

(a choice of such $\psi, f$ is possible thanks to (3.1)) we have by a partition of unity and Lemma 3.4

$$\mathrm{Tr}^\flat\Big(f\int\psi(A)e^{-X_A}\,dA\Big) = \sum_{A\in\mathcal{W}\cap L(T_{x_0})}\frac{\psi(A)\int_{T_{x_0}} f}{|\det(1-\mathcal{P}_A)|}.$$

Consequently, using (3.1), since the measure on the orbits is given by the push-forward of the Lebesgue measure on $\mathbb{A}$, the formula is thus established for compactly supported functions.

The remaining **third step** is to consider the convergence of the formula when the support is non-compact. This follows from two observations. The first one is that we have an exponential bound on the number of closed orbits

**Lemma 3.5.** *Let $n = \dim\mathcal{M}$, let $dA$ be the Haar measure on $\mathfrak{a}$ and $M = \sup_{A\in\mathcal{W},|A|=1}\|\varphi_1^A\|_{\mathcal{L}(C^2(\mathcal{M},\mathbb{R}))}$. Then there is $C>0$ such that for all $\ell\geq 0$*

$$\sharp\{A\in L(T)\mid T\in\mathcal{T},\,|A|\leq\ell\}\leq C\ell^\kappa e^{(n-\kappa)M\ell},$$
$$\sharp\{T\in\mathcal{T}\mid \exists A\in L(T), |A|\leq\ell\}\leq C\ell^\kappa e^{(n-\kappa)M\ell}$$

*and for each $\epsilon>0$ and $\delta>0$,*

$$(v_g\otimes dA)(\{(x,A)\in\mathcal{M}\times\mathcal{W}\mid |A|\in(\delta,\ell), d_g(x,\varphi_1^A(x))<\epsilon\})\leq C\epsilon^n e^{nM\ell}. \tag{3.5}$$

The proof of this lemma is the object of Appendix A. The second observation is that given a closed cone $\mathcal{C}\subset\mathcal{W}$, there exists $C>0$ such that for $T\in\mathcal{T}$, $A\in L(T)\cap\mathcal{C}$, and $x\in T$,

$$\frac{1}{|\det(1-\mathcal{P}_A(x))|}\leq Ce^{C|A|}.$$

This follows from the fact that the hyperbolicity constants in formula (1.3) and (1.4) can only degenerate at the boundary of $\mathcal{W}$. □

We can apply the Guillemin trace formula to our integrated propagators. Given $\psi\in C_c^\infty(\mathcal{W})$ with $\int\psi = 1$ and $\mathrm{supp}(\psi)$ contained in a small neighborhood of an element $A_0\in\mathcal{W}$, and $f\in C^\infty(\mathcal{M})$, we obtain

$$\mathrm{Tr}^\flat(fR_\psi(\lambda)^k) = \sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)}\frac{\int_T f}{|\det(1-\mathcal{P}_A)|}e^{-\lambda(A)}\psi^{(k)}(A) \tag{3.6}$$

where $\psi^{(k)}$ is the $k$-th convolution power of $\psi$.

3.2. **Flat trace of the shifted resolvent.** The purpose of this section is to study the shifted resolvent, and its flat trace. That is to say that for $f\in C^\infty(\mathcal{M}), \psi\in C_c^\infty(\mathcal{W})$ and $\lambda\in\mathfrak{a}_{\mathbb{C}}^*, s\in\mathbb{C}$, we will consider

$$Z_{\psi,f}(s,\lambda) := \mathrm{Tr}^\flat(fR_\psi(\lambda)(s-R_\psi(\lambda))^{-1}) \tag{3.7}$$

and prove the:

**Proposition 3.6.** *Let $\tau : \mathbb{A} \to \mathrm{Diffeo}(\mathcal{M})$ be an Anosov action with Weyl chamber $\mathcal{W}$. For $f \in C^\infty(\mathcal{M})$, and $\psi \in C^\infty_c(\mathcal{W})$, with support small enough, the function $Z_{\psi,f}$ originally defined for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $\mathrm{Re}\,\lambda$ large enough, and $s \in \mathbb{C}$ with $|s-1| < 1/2$ has a meromorphic extension to $\mathfrak{a}^*_{\mathbb{C}} \times B_{\mathbb{C}}(1, 1/2)$. Moreover, we have the formula:*

$$Z_{f,\psi}(s, \lambda) = \sum_{k=1}^{\infty} s^{-k} \sum_{T\in\mathcal{T}} \sum_{A\in\mathcal{W}\cap L(T)} \frac{\int_T f}{|\det(1 - \mathcal{P}_A)|} e^{-\lambda(A)} \psi^{(k)}_\sigma(A). \tag{3.8}$$

*Finally, if we replace $\psi$ by $\psi_\sigma := \psi(\cdot - \sigma)$, then $Z_{f,\psi_\sigma}$ depends continuously on $\sigma$ in a small neighbourhood of $0 \in \mathfrak{a}^*$. The topology on $Z_{f,\psi_\sigma}$ is given by uniform convergence on compact subsets of the holomorphic regions in $\mathfrak{a}^*_{\mathbb{C}} \times B_{\mathbb{C}}(1, 1/2)$.*

The proof of this theorem will follow the ideas of the proof by Dyatlov and Zworski of the meromorphic extension of the dynamical determinant of Anosov flows [DZ16]. The idea is to use propagation of singularities, and source/sink estimates to control the wavefront set of the resolvent. We will explain this in detail. If $\Gamma \subset T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\}$ is a conic closed set, define

$$C^{-\infty}_\Gamma(\mathcal{M} \times \mathcal{M}) := \{u \in C^{-\infty}(\mathcal{M} \times \mathcal{M}) \mid \mathrm{WF}(u) \subset \Gamma\}$$

the space of distributions on $\mathcal{M}\times\mathcal{M}$ with wave-front set included in $\Gamma$. Its topology is defined using sequences in [Hör03, Definition 8.2.2].

To analyze the wave-front set of the resolvent of $R_\psi(\lambda)$, it will be convenient to work with a small semiclassical parameter $h > 0$. We also introduce a semiclassical quantization $\mathrm{Op}_h$ and define $\mathcal{H}^{NG}_h = \mathrm{Op}_h(e^{NG})^{-1} L^2(\mathcal{M})$. As a vector space, $\mathcal{H}^{NG}_h$ is equal to $\mathcal{H}^{NG}$ ; only the norm is different. We will denote by $\Psi^m_h(\mathcal{M})$ the space of semiclassical pseudo-differential operators of order $m \in \mathbb{R}$ (see [Zwo12] or [DZ19, Appendix E]). We recall briefly that $Q \in \Psi^m_h(\mathcal{M})$ can be written as $Q = \mathrm{Op}_h(q) + Q'$ with $Q'$ an operator having smooth Schwartz kernel with its $C^k$ norms being $\mathcal{O}(h^\infty)$ for all $k \in \mathbb{N}$ and $q \in S^m(T^*\mathcal{M})$ a symbol of order $m$, and $\mathrm{WF}_h(Q) \subset T^*\mathcal{M}$ is the complement to the set of points where $q$ and its derivatives is equal to $\mathcal{O}(h^\infty(1 + |\xi|)^{-\infty})$ ([DZ19, Definition E.26]). We shall also use the notation $\overline{T}^*\mathcal{M}$ for the radially compactified cotangent bundle and semiclassical wave-front sets can be viewed as closed subsets of $\overline{T}^*\mathcal{M}$ (see [DZ19, Appendix E]). We denote by $\Psi^{\mathrm{comp}}_h(\mathcal{M})$ the space of those semiclassical pseudo-differential operators with compact semiclassical wave-front set. Below, we say that a family $\theta \in \mathbb{R}^n \mapsto \mathcal{K}_\theta \in C^{-\infty}(\mathcal{M} \times \mathcal{M})$ of Schwartz kernels of operators has wave-front set contained in $\Gamma \subset T^*(\mathcal{M} \times \mathcal{M})$ locally uniformly in $\tau$ if for each closed conic subset $\Omega \in T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\}$ not intersecting $\Gamma$, and each $B, B' \in \Psi^0(\mathcal{M})$ satisfying $\mathrm{WF}(B) \times \mathrm{WF}(B') \subset \{(x, \xi, x', -\xi') \,|\, (x, \xi, x', \xi') \in \Omega\}$, for each $N \in \mathbb{N}$ and each compact set $K \subset \mathbb{R}^n$, there is $C_{N,K,B,B'} > 0$ so that for all $\theta \in K$

$$\|B\mathcal{K}_\theta B'\|_{\mathcal{L}(H^N(\mathcal{M}),H^{-N}(\mathcal{M}))} \leq C_{N,K,B,B'}.$$

**Proposition 3.7.** *Let $\mathcal{O}$ be a small enough convex open set, relatively compact in $\mathcal{W}$, let $\psi \in C^\infty_c(\mathcal{O}; \mathbb{R}^+)$ and define $\psi_\sigma = \psi(\cdot - \sigma)$ for $\sigma \in \mathfrak{a}$ small so that $\psi_\sigma \in C^\infty_c(\mathcal{O})$. Then the operator $T_{\psi_\sigma}(\lambda, s) := R_{\psi_\sigma}(R_{\psi_\sigma}(\lambda) - s)^{-1} : \mathcal{H}^{NG} \to \mathcal{H}^{NG}$ is analytic in $|s| > 1$ and meromorphic in $|s| > e^{-c_0 N}$ for some $c_0 > 0$ and all $N > 0$. Moreover, locally uniformly in $(\sigma, \lambda, s)$ (where it is defined) the Schwartz kernel $\mathcal{K}_{T_{\psi_\sigma}(\lambda,s)}$ of $R_{\psi_\sigma}(\lambda)(R_{\psi_\sigma}(\lambda) - s)^{-1}$ has wave-front set contained in*

$$\mathrm{WF}(\mathcal{K}_{T_\psi(s)}) \subset \left\{ (e^{X^H_A}(x, \xi), (x, \xi)) \,\middle|\, (x, \xi) \in E^*_u \oplus E^*_s,\, A \in k\mathcal{O},\, k \in \mathbb{N} \setminus \{0\} \right\} \cup (E^*_u \times E^*_s)$$

*where $X_A^H$ is the Hamilton flow of $\xi(X_A(x))$, the principal symbol of $-iX_A$ for $A \in \mathcal{W}$. There is a cone $\Gamma \subset T^*(\mathcal{M}\times\mathcal{M})$ such that $\Gamma \cap N^*\Delta = \emptyset$ and*

$$(\sigma, \lambda, s) \mapsto \mathcal{K}_{T_{\psi_\sigma}(\lambda,s)} \in C_\Gamma^{-\infty}(\mathcal{M}\times\mathcal{M})$$

*is continuous where it is defined.*

This is the main technical result; it is similar to [DZ16, Proposition 3.3]. Using the tools of [Hör03, Chapter 8], we deduce directly that $Z_{f,\psi}$ is well-defined as a meromorphic function (with the continuous dependence on $\sigma$ for $Z_{f,\psi_\sigma}$). It will remain to obtain Formula (3.8) to prove Proposition 3.6.

To prove Proposition 3.7, we will rely on a wavefront set estimate for a parametrix – much as [DZ16]. This is in some sense a refinement of [GBGHW20, Lemma 4.14]:

**Lemma 3.8.** *If $\mathcal{O}$ is small enough and $A_0 \in \mathcal{O}$, there exists $c_0, c_1 > 0$ and $Q \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ such $R_\psi(\lambda)(1-Q) - s$ has a bounded inverse on $\mathcal{H}_h^{NG}$ for $|s| > e^{-c_0 N}$ and $\operatorname{Re}\lambda(A_0) > -c_1 N$, $|\operatorname{Im}\lambda| < h^{-1/2}$ for all $N > 0$. In that region, its inverse $T_{\psi_\sigma}^Q(\lambda, s) := (R_{\psi_\sigma}(\lambda)(1-Q) - s)^{-1}$ is an analytic family of bounded operator in $(\lambda, s)$ and its Schwartz kernel $\mathcal{K}_{T_{\psi_\sigma}^Q(\lambda,s)}$ satisfies uniformly in $(\sigma, \lambda, s)$*

$$\mathrm{WF}_h(\mathcal{K}_{T^Q_{\psi_\sigma}(s)}) \cap T^*(\mathcal{M}\times\mathcal{M}) \subset N^*\Delta \cup \Omega_+(\mathcal{O}) \tag{3.9}$$

$$\Omega_+(\mathcal{O}) := \bigcup_{k\geq 1} \Omega_+^k(\mathcal{O}), \quad \Omega_+^k(\mathcal{O}) := \left\{ (e^{X_A^H}(x,\xi), (x, -\xi)) \;\middle|\; (x,\xi) \in E_u^* \oplus E_s^*,\ A \in k\mathcal{O} \right\}.$$

*Proof.* We start with the proof of existence of $T_\psi^Q$. For this, we pick $\Gamma_{E_0^*} \subset T^*\mathcal{M}$, a conic neighborhood of $E_0^*$, and then $G \in C^\infty(T^*\mathcal{M})$ an escape function for all $A \in \mathcal{O}$ compatible with $c_X > 0$ and $\Gamma_{E_0^*} \subset T^*\mathcal{M}$ in the sense of [GBGHW20, Definition 4.1]. This is possible, because we can take such a function adapted to $A_0 \in \mathcal{O}$, and it remains adapted to all $A$'s sufficiently close to $A_0$. (This is the first reason for assuming that $\mathcal{O}$ is small enough). By the properties (1.21) of the escape function, there is $c_X > 0$ such that for $A \in \mathcal{O}$, and some $r > 0$ large enough,

$$\bigcup_{t\in[0,1]} e^{tX_A^H}(x,\xi) \cap \Gamma_{E_0^*} = \emptyset,\ |\xi| > r \Longrightarrow G(e^{X_A^H}(x,\xi)) - G(x,\xi) \leq -c_X.$$

Since it will be used several times below, notice that for $\psi \in C_c^\infty(\mathcal{O})$, using the convexity of $\mathcal{O}$,

$$\mathrm{supp}(\psi^{(k)}) \subset \mathrm{supp}(\psi) + \cdots + \mathrm{supp}(\psi) \subset k\mathcal{O}.$$

This is contained in a ball of radius $\delta k$ centered at $kA_0$ for some small $\delta$, given that $\mathcal{O}$ is small. We imitate now the proof of [GBGHW20, Lemma 4.5] but with a semiclassical quantization. The idea here is that in the direction of the flow $E_0^*$, averaging the flow is regularizing, and in the transverse direction, we can use the escape function to obtain some compactness. Let $\Gamma_0 := \Gamma_{E_0^*} \cap \{|\xi| \geq 1\}$ and choose $P \in \Psi_h^0(\mathcal{M})$ which satisfies

$$\mathrm{WF}_h(P) \subset \left\{ (x,\xi) \in \overline{T}^*\mathcal{M} \;\middle|\; \forall t \in [0,1],\ e^{tX_A^H}(x,\xi) \notin \Gamma_0 \right\}$$

and $0 \leq \sigma(P) \leq 1$. The operator $P$ is microlocalizing away from the neutral direction $E_0^*$. We also pick $\Gamma_0'$ a neighbourhood of $\Gamma_0$ which is conic for $|\xi| \geq 1$ and contained in $|\xi| > 1/2$, and assume that

$$\mathrm{WF}_h(1-P) \subset \Gamma_0'$$

(i.e. $P$ is microlocally equal to 1 outside $\Gamma_0$.) Note that we can chose $\Gamma_0'$ such that $T^*\mathcal{M}\setminus\Gamma_0$ is for $|\xi|>1$ an arbitrary small cone around $E_u^*\oplus E_s^*$. Setting for $A\in\mathcal{O}$, $B_A:=e^{X_A}\operatorname{Op}_h(e^{NG})e^{-X_A}$, we have

$$\operatorname{Op}_h(e^{NG})e^{-X_A}P\operatorname{Op}_h(e^{NG})^{-1}=e^{-X_A}B_AP\operatorname{Op}_h(e^{NG})^{-1}.$$

The semi-classical principal symbol of $B_AP\operatorname{Op}_h(e^{NG})^{-1}\in\Psi_h^0(\mathcal{M})$ is (by Egorov's Lemma)

$$\sigma(B_AP\operatorname{Op}_h(e^{NG})^{-1})=e^{N(G\circ e^{X_A^H}-G)}\sigma(P)\mod hS^{-1}.$$

Using the properties (1.21) of the escape function $G$, we find for $r>0$ large enough and some $c_X>0$

$$\sup_{|\xi|\geq r}|\sigma(B_AP\operatorname{Op}_h(e^{NG})^{-1})(x,\xi)|\leq e^{-c_XN}.$$

Next, we introduce $Q_0\in\Psi_h^{\mathrm{comp}}(\mathcal{M})$ so that $\mathrm{WF}_h(Q_0)\subset\{|\xi|\leq 2r\}$ and $\mathrm{WF}_h(1-Q_0)\subset\{|\xi|\geq r\}$ with $0\leq\sigma(Q_0)\leq 1$. We also let $C_{\mathcal{O}}:=\max_{A\in\mathcal{O}}\|e^{-X_A}\|_{\mathcal{L}(L^2(\mathcal{M}))}$. The previous estimate implies that for all $h>0$ small and $A\in\mathcal{O}$,

$$\|e^{-X_A}P(1-Q_0)\|_{\mathcal{L}(\mathcal{H}_h^{NG})}\leq C_{\mathcal{O}}e^{-c_XN}+\mathcal{O}(h). \tag{3.10}$$

We used $[Q_0,\operatorname{Op}_h(e^{NG})^{-1}]\operatorname{Op}_h(e^{NG})\in h\Psi_h^{-1}(\mathcal{M})$. The operator $e^{-X_A}PQ_0$ is compact and smoothing. We get (using $\int\psi=1$)

$$\|R_\psi(\lambda)P(1-Q_0)\|_{\mathcal{L}(\mathcal{H}_h^{NG})}\leq C_{\mathcal{O}}e^{-c_XN}\sup_{A\in\mathcal{O}}e^{-\operatorname{Re}\lambda(A)}+\mathcal{O}(h).$$

We now recall how the smoothing effect in the direction of the flow appears. As in the proof of [GBGHW20, Lemma 4.14], for each $A_1,\dots,A_m\in\mathcal{W}$ we see by integration by parts that

$$R_\psi(\lambda)(X_{A_1}+\lambda(A_1))\dots(X_{A_m}+\lambda(A_m))=\int_{\mathcal{W}}(\partial_{A_1}\dots\partial_{A_m}\psi(A))e^{-X_A-\lambda(A)}dA.$$

Let us pick $A_1,\dots,A_\kappa\in\mathcal{W}$ a local basis and let

$$\Delta^0_{\mathbb{A}}(\lambda):=-\sum_{j=1}^{\kappa}(\partial_{A_j}+\lambda(A_j))^2,\ \Delta_{\mathbb{A}}(\lambda):=\tau_*\Delta^0_{\mathbb{A}}(\lambda)=-\sum_{j=1}^{\kappa}(X_{A_j}+\lambda(A_j))^2.$$

(The first one acts on $\mathbb{A}$ while the second acts on $\mathcal{M}$.) Since we assumed that $|\operatorname{Im}\lambda|<h^{-1/2}$, $h^2\Delta_{\mathbb{A}}(\lambda)$ is elliptic on the wavefront set of $1-P$, uniformly in $\lambda$. We can thus find uniformly in $\lambda$ for each $m\in\mathbb{N}$ a parametrix $S(\lambda)\in\Psi_h^{-2m}(\mathcal{M})$ so that

$$(h^2\Delta_{\mathbb{A}}(\lambda))^mS(\lambda)(1-P)-(1-P)\in h^\infty\Psi_h^{-\infty}(\mathcal{M})$$

(actually, $S(\lambda)$ is a holomorphic function of $h\lambda$). We thus deduce that

$$R_\psi(\lambda)(1-P)-h^{2m}\int_{\mathcal{W}}[\Delta^0_{\mathbb{A}}(\lambda)^m\psi_\sigma](A)e^{-X_A-\lambda(A)}S(\lambda)(1-P)\,dA\in h^\infty\Psi_h^{-\infty}(\mathcal{M}).$$

(here the bound on the remainder does not depend on $\psi$ since $\|\psi\|_{L^1}=1$.) In particular, for each $m\geq 0$ there is $C_m>0$ depending continuously on $\|D^{2m}\psi\|_{L^1(\mathcal{O})}$ such that

$$\|R_\psi(\lambda)(1-P)\|_{\mathcal{L}(\mathcal{H}_h^{NG})}\leq C_mh^{2m}. \tag{3.11}$$

The same argument (using the operator $\Delta_{\mathbb{A}}(\lambda)$ acting on the other side) shows that

$$\|(1-P)R_\psi(\lambda)\|_{\mathcal{L}(\mathcal{H}_h^{NG})}\leq C_mh^{2m}. \tag{3.12}$$

We thus conclude that there is $c_0 > 0$ and $c_1 > 0$ such that for all $m > 0$, all $h > 0$ small enough and $\mathrm{Re}\lambda(A_0) > -c_1 N$ and $|\mathrm{Im}(\lambda)| < h^{-1/2}$

$$\begin{gathered} R_\psi(\lambda) = R_\psi^0(\lambda) + R_\psi^1(\lambda) + R_\psi(\lambda)Q, \\ R_\psi^0(\lambda) := R_\psi(\lambda)P(1-Q_0), \quad R_\psi^1(\lambda) := R_\psi(\lambda)(1-P), \quad Q := PQ_0, \\ \|R_\psi^0(\lambda)\|_{\mathcal{L}(\mathcal{H}_h^{NG})} \le e^{-c_0 N}, \quad \|R_\psi^1(\lambda)\|_{\mathcal{L}(\mathcal{H}_h^{NG})} \le C_m h^m. \end{gathered} \tag{3.13}$$

This shows that $R_\psi(\lambda)(1-Q) - s$ is invertible on $\mathcal{H}_h^{NG}$ for $|s| > e^{-c_0 N}$ and $h > 0$ small enough, locally uniformly in $\psi$ and $\lambda$. We call $T_\psi^Q(\lambda, s)$ its inverse.

The second step is to prove the announced property of its wavefront set. We will assume that $\psi_\sigma = \psi(\cdot - \sigma)$ is a family of $C_c^\infty(\mathcal{O})$ functions for some parameter $\sigma \in \mathfrak{a}^*$ small. We thus have a family $T_{\psi_\sigma}^Q(\lambda, s)$ of operators depending on $\theta := (\lambda, s, \sigma)$ and we will assume it lives in a small compact set $K$ where $T_{\psi_\sigma}^Q(\lambda, s)$ is well-defined.

We start by recalling a few basic facts about wave-front sets for families of operators and their Schwartz kernels; we refer to [DZ16, Lemma 2.3] or [DGRS20, Lemma 6.2] for the parameter dependent version. A point $(x, \xi, x', -\xi') \in T^*(\mathcal{M} \times \mathcal{M})$ is not in $\mathrm{WF}_h(\mathcal{K}_{T_{\psi_\sigma}^Q(\lambda,s)})$ uniformly in $\theta \in K$ if there are $\theta$-independent neighborhoods $U$ of $(x', \xi')$ in $T^*\mathcal{M}$ and $V$ of $(x, \xi)$ such that for any $B, B' \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ with $\mathrm{WF}_h(B) \subset V$ and $\mathrm{WF}_h(B') \subset U$, for any $m \ge 0$ there is $C_m$ so that for all $h > 0$ small and all $\theta \in K$

$$\|BT_{\psi_\sigma}^Q(\lambda, s)B'\|_{\mathcal{L}(L^2)} \le C_m h^m.$$

For notational simplicity we shall say that the RHS is an $\mathcal{O}(h^\infty)$ uniformly in $\theta \in K$.

Let us start with the elliptic region. We observe that

$$\begin{aligned} T_{\psi_\sigma}^Q(\lambda, s) &= -\frac{1}{s}\left(1 - \frac{R_{\psi_\sigma}(\lambda)(1-Q)}{s}\right)^{-1} \\ &= -\frac{1}{s} - \frac{R_{\psi_\sigma}(\lambda)(1-Q)}{s^2} - \frac{R_{\psi_\sigma}(\lambda)(1-Q)}{s^3} T_{\psi_\sigma}^Q(\lambda, s) R_{\psi_\sigma}(\lambda)(1-Q). \end{aligned} \tag{3.14}$$

Applying respectively $(1-P)$ on the right and then on the left (for $P$ chosen as above with a $\Gamma_0'$ coming arbitrary cloese to $E_u^* \oplus E_s^*$) and using that $\mathrm{WF}_h(1-Q) \subset \{|\xi| \ge r\}$, we obtain using (3.11) and (3.12) that there is $r' \in (0, r)$ such that uniformly in $\theta \in K$,

$$\begin{aligned} &\mathrm{WF}_h(\mathcal{K}_{T_{\psi_\sigma}^Q(\lambda,s)}) \ \cap \ T^*(\mathcal{M} \times \mathcal{M}) \subset \\ &\qquad N^*\Delta \ \cup \left\{(x, \xi, x', \xi') \ \middle|\ (x, \xi) \in E_u^* \oplus E_s^* \text{ and } (x', \xi') \in E_u^* \oplus E_s^*, \text{ and } |\xi|, \ |\xi'| \ge r'\right\}. \end{aligned} \tag{3.15}$$

It remains to consider the intersection of the wavefront set with $(E_u^* \oplus E_s^*)^2$. Expanding the formula for $T_{\psi_\sigma}^Q(\lambda, s)$, we get a convergent sum in $\mathcal{L}(\mathcal{H}_h^{NG})$

$$T_{\psi_\sigma}^Q(\lambda, s) = -\frac{1}{s} \sum_{k \ge 0} \left(\frac{R_{\psi_\sigma}(\lambda)(1-Q)}{s}\right)^k.$$

This is the inspiration for the wavefront set statement. However, since the terms in the sum are not increasingly smoothing, only smaller and smaller, we cannot directly obtain the desired statement.

Let us rewrite exactly what we need to prove. For each $B, B' \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ microsupported near $(E_u^* \oplus E_s^*) \cap \{|\xi| \geq r'\}$, so that $\mathrm{WF}_h(B') \cap \Omega_B = \emptyset$ with

$$\Omega_B := \left\{(x,\xi) \;\middle|\; e^{X_A^H}(x,\xi) \in \mathrm{WF}_h(B),\ A \in \cup_{k\geq 0} k\mathcal{O}\right\},$$

it suffices to prove that uniformly for $\theta \in K$

$$\|BT^Q_{\psi_\sigma}(\lambda, s)B'\|_{\mathcal{L}(L^2)} = \mathcal{O}(h^\infty) \tag{3.16}$$

Note that the conormal region $N^*\Delta$ is covered because we now include $k = 0$ in the definition of $\Omega_B$.

It will be convenient below to use that for $|s| > e^{-c_0 N}$, if $u = T^Q_{\psi_\sigma}(\lambda, s)f$ one has $u = (-f + R^0_{\psi_\sigma}(\lambda)u + R^1_{\psi_\sigma}(\lambda)u)/s$, thus for each $B \in \Psi^0_h(\mathcal{M})$ with $|\sigma(B)| \leq 1$, we have uniformly for $\theta \in K$

$$\|Bu\|_{\mathcal{H}_h^{NG}} \leq |s|^{-1}(\|Bf\|_{\mathcal{H}_h^{NG}} + \|BR^0_{\psi_\sigma}(\lambda)u\|_{\mathcal{H}_h^{NG}}) + \mathcal{O}(h^\infty). \tag{3.17}$$

First we will need a so-called *source estimate* close to $E_s^* \cap \partial\overline{T}^*\mathcal{M}$ in order to control the wave-front set in that region.

**The source estimate.** Let $V_s \subset \overline{T}^*\mathcal{M}$ be a small neighborhood of $L := E_s^* \cap \partial\overline{T}^*\mathcal{M}$ so that $e^{-X_A^H}(V_s) \subset V_s$ for all $A \in \mathcal{O}$, $m(x,\xi)$ is constant in $V_s$ and $P = 1$ microlocally on $V_s$. Now using the fact that $L$ is a repulsor (a source) for the flow $e^{X_A^H}$ with $A \in \mathcal{O}$, there is $b_1, b_2 \in S^0(T^*\mathcal{M})$ with $\mathrm{supp}(b_2) \subset \{b_1 = 1\}$ so that $b_2 \circ e^{-X_A^H} = 1$ on $\mathrm{supp}(b_1)$ for all $A \in \mathcal{O}$. Notice that for $v \in \mathcal{H}_h^{NG}$ and $B_{1/2} = \mathrm{Op}_h(b_{1/2})$

$$\|B_2 v\|_{\mathcal{H}_h^{NG}} \leq \|B_1 B_2 v\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty \|v\|_{\mathcal{H}_h^{NG}}).$$

This means that, thanks to the Egorov's Lemma, $B_1 R^0_{\psi_\sigma}(\lambda) - B_1 R^0_{\psi_\sigma}(\lambda) B_2 \in h^\infty \Psi_h^{-\infty}(\mathcal{M})$. The remainder here depends continuously on $\sigma, \lambda$ as before. Thus, using (3.13) we get for all $v \in C^\infty(\mathcal{M})$

$$\begin{aligned}\|B_1 R^0_\psi(\lambda) v\|_{\mathcal{H}_h^{NG}} &\leq \|B_1 R^0_\psi(\lambda) B_2 v\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty \|v\|_{\mathcal{H}_h^{NG}}),\\ &\leq e^{-c_0 N}\|B_2 v\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty \|v\|_{\mathcal{H}_h^{NG}}),\\ &\leq e^{-c_0 N}\|B_1 v\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty \|v\|_{\mathcal{H}_h^{NG}}).\end{aligned}$$

Combining with (3.17), and assuming $e^{-c_0 N}|s|^{-1} \leq q < 1$,this yields that there is $C > 0$ such that for all $f \in C^\infty(\mathcal{M})$

$$\|B_1 u\|_{\mathcal{H}_h^{NG}} = \|B_1 T^Q_{\psi_\sigma}(\lambda, s) f\|_{\mathcal{H}_h^{NG}} \leq C\|B_1 f\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty \|f\|_{\mathcal{H}_h^{NG}}) \tag{3.18}$$

uniformly for $(\sigma, \lambda, s) \in K$.

2) **Propagation estimate outside $E_u^*$.** We will next show how to use the source estimate to obtain information on the wave-front set of $\mathcal{K}_{T_{\psi_\sigma}(\lambda,s)}$ outside $E_u^*$.

Assume that $B \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ satisfies $\mathrm{WF}_h(B) \cap E_u^* = \emptyset$, $0 \leq \sigma(B) \leq 1$ and $\mathrm{WF}_h(B)$ contained in a small neighborhood of $E_u^* \oplus E_s^*$ and $B' \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ satisfying $\mathrm{WF}_h(B') \cap \Omega_B = \emptyset$. Since $L$ is an repulsor, there is $k \in \mathbb{N}$ large enough such that for all $A \in k\mathcal{O}$, $e^{-X_A^H}(\mathrm{WF}_h(B)) \subset V_s$ and $\mathrm{WF}_h(B') \cap V_s = \emptyset$ for some small enough neighborhood $V_s$ of $L$, invariant by $e^{-X_A^H}$ as in 1). We will use $B_1$ as in 1) for this set $V_s$ and we can assume, up to taking $k$ even larger, that $e^{-X_A^H}(\mathrm{WF}_h(B)) \subset \mathrm{supp}(b_1)$ for all $A \in k\mathcal{O}$.

Iterating (3.17), we obtain for each $k \geq 1$ and each $f \in C^\infty(\mathcal{M})$ (with $u = T^Q_{\psi_\sigma}(\lambda, s)f$)

$$\|Bu\|_{\mathcal{H}_h^{NG}} \leq (\sum_{j=0}^{k-1} |s|^{-j-1}\|B(R^0_{\psi_\sigma}(\lambda))^j f\|_{\mathcal{H}_h^{NG}} + |s|^{-1}\|B(R^0_{\psi_\sigma}(\lambda))^k u\|_{\mathcal{H}_h^{NG}}) + \mathcal{O}(h^\infty\|f\|_{\mathcal{H}_h^{NG}}). \tag{3.19}$$

By Egorov theorem, locally uniformly in $(\sigma, \lambda, s)$

$$\mathrm{WF}_h(B(R^0_{\psi_\sigma}(\lambda))^j f) \subset \mathrm{WF}_h(B) \cap \bigcup_{A\in j\mathcal{O}} e^{X_A^H}(\mathrm{WF}_h(f)).$$

Again by Egorov, we have $B(R^0_{\psi_\sigma}(\lambda))^k - B(R^0_{\psi_\sigma}(\lambda))^k B_1 \in h^\infty \Psi_h^\infty(\mathcal{M})$ thus by (3.18)

$$\|BT^Q_{\psi_\sigma}(\lambda, s)f\|_{\mathcal{H}_h^{NG}} \leq C(\sum_{j=0}^{k-1} |s|^{-j-1}\|B(R^0_{\psi_\sigma}(\lambda))^j f\|_{\mathcal{H}_h^{NG}} + |s|^{-1}\|B_1 f\|_{\mathcal{H}_h^{NG}}) + \mathcal{O}(h^\infty\|f\|_{\mathcal{H}_h^{NG}}).$$

with $C > 0$ and the remainder being locally uniform in $(\sigma, \lambda, s)$. Applying this with $B'f$ instead of $f$ and using $B_1 B' \in h^\infty \Psi_h^{-\infty}(\mathcal{M})$ and $B(R^0_{\psi_\sigma}(\lambda))^j B' \in h^\infty \Psi_h^{-\infty}(\mathcal{M})$ for $j \in [0, k]$ under our assumptions on $B, B'$, we obtain (3.16) uniformly in $(\sigma, \lambda, s) \in K$.

**3) Estimate near $E_u^*$.** As we already have the estimate (3.15) on $\mathrm{WF}(\mathcal{K}_{T^Q_{\psi_\sigma}})$ there remains to study the case where $\mathrm{WF}_h(B)$ and $\mathrm{WF}_h(B')$ are intersecting $E_u^*$. By Egorov and the fact that $u := T^Q_{\psi_\sigma}(\lambda, s)B'f$ has $\mathrm{WF}_h(u)$ contained in $\{|\xi| \geq r'\}$ (which can be read off (3.14)), we obtain that $\mathrm{WF}_h(R^0_{\psi_\sigma}(\lambda)u) \subset \cup_{A\in\mathcal{O}} e^{X_A^H}(\mathrm{WF}_h(u))$ is contained in $\{|\xi| \geq r'/C_0\}$ for some $C_0 > 0$. We can assume that $\mathrm{WF}_h(B)$ is a small neighborhood of a point $(x, \xi) \in E_u^* \cap \{|\xi| \geq r'\}$, so that for $k \geq 2$ large enough $e^{-X_A^H}(\mathrm{WF}_h(B)) \in \{|\xi| \leq r'/2C_0\}$ for all $A \in (k-1)\mathcal{O}$. We then obtain by Egorov that $\|B(R^0_{\psi_\sigma}(\lambda))^{k-1}R^0_{\psi_\sigma}(\lambda)u\|_{\mathcal{H}_h^{NG}} = \mathcal{O}(h^\infty)$ uniformly in $\theta \in K$. The estimate (3.19) still holds and we deduce that uniformly for $\theta \in K$

$$\|BT^Q_{\psi_\sigma}(\lambda, s)B'f\|_{\mathcal{H}_h^{NG}} \leq \sum_{j=0}^{k-1} |s|^{-j-1}\|B(R^0_{\psi_\sigma}(\lambda))^j B'f\|_{\mathcal{H}_h^{NG}} + \mathcal{O}(h^\infty\|f\|_{\mathcal{H}_h^{NG}}).$$

Assuming that $\mathrm{WF}_h(B') \cap \Omega_B = \emptyset$, the right hand side is an $\mathcal{O}(h^\infty)$ uniformly for $\theta \in K$ by Egorov theorem again.

We conclude that (3.9) holds. ☐

We can now turn to

*Proof of Proposition 3.7.* Next we write $(R_{\psi_\sigma}(\lambda) - s)T^Q_{\Psi_\sigma}(\lambda, s) = 1 - R_{\psi_\sigma}(\lambda)QT^Q_{\Psi_\sigma}(\lambda, s)$ and $T^Q_{\Psi_\sigma}(\lambda, s)(R_{\psi_\sigma}(\lambda) - s) = 1 - T^Q_{\Psi_\sigma}(\lambda, s)R_{\psi_\sigma}(\lambda)Q$ for $|s| > e^{-c_0 N}$ on $\mathcal{H}_h^{NG}$. Denote by $Q' := R_{\psi_\sigma}(\lambda)Q \in \Psi_h^{\mathrm{comp}}(\mathcal{M})$ which is compact on $\mathcal{H}_h^{NG}$. We can then use Fredholm theorem to show that $(1 - Q'T^Q_{\Psi_\sigma}(\lambda, s))^{-1}$ extends meromorphically in $s \in \{|s| > e^{-c_0 N}\}$ on $\mathcal{H}_h^{NG}$ and

$$(R_{\psi_\sigma}(\lambda) - s)^{-1} = T^Q_{\Psi_\sigma}(\lambda, s)(1 - Q'T^Q_{\Psi_\sigma}(\lambda, s))^{-1}$$

is a meromorphic extension of $(R_{\psi_\sigma}(\lambda) - s)^{-1} \in \mathcal{L}(\mathcal{H}_h^{NG})$ to the region $\{|s| > e^{-c_0 N}\}$. This also implies that for $|s| \gg 1$

$$(R_{\psi_\sigma}(\lambda) - s)^{-1} = T^Q_{\Psi_\sigma}(\lambda, s) + T^Q_{\Psi_\sigma}(\lambda, s)Q'T^Q_{\Psi_\sigma}(\lambda, s) + T^Q_{\Psi_\sigma}(\lambda, s)Q'(R_{\psi_\sigma}(\lambda) - s)^{-1}Q'T^Q_{\Psi_\sigma}(\lambda, s).$$

By Lemma 3.8 and since $R_{\psi_\sigma}(\lambda)$ is $h$-independent, we have

$$\mathrm{WF}_h(T^Q_{\psi_\sigma}(\lambda,s)Q'T^Q_{\psi_\sigma}(\lambda,s))\cap T^*(\mathcal{M}\times\mathcal{M})\subset\Upsilon,$$
$$\mathrm{WF}_h(T^Q_{\psi_\sigma}(\lambda,s)Q'(R_{\psi_\sigma}(\lambda)-s)^{-1}Q'T^Q_{\psi_\sigma}(\lambda,s))\cap T^*(\mathcal{M}\times\mathcal{M})\subset\Upsilon,$$
$$\Upsilon:=\{(x,\xi,x',-\xi')\in T^*(\mathcal{M}\times\mathcal{M})\,|\,(x,\xi)\in E_u^*\oplus E_s^*,\,\exists k,k'\in\mathbb{N},\exists A\in k\mathcal{O},$$
$$\exists A'\in k'\mathcal{O}, e^{-X_A^H}(x,\xi)\in\mathrm{WF}_h(Q'), e^{X^H_{A'}}(x',\xi')\in\mathrm{WF}_h(Q')\}$$

and this holds uniformly in $\theta\in K$. Since $\mathrm{WF}_h(R_{\psi_\sigma}(\lambda))\subset\Omega^1_+$, this shows that

$$\mathrm{WF}_h(R_{\psi_\sigma}(\lambda)(R_{\psi_\sigma}(\lambda)-s)^{-1})\cap T^*(\mathcal{M}\times\mathcal{M})\subset\Omega_+(\mathcal{O})\cup\Upsilon.$$

Now, using that $\mathrm{WF}_h(Q')$ is a compact set, we observe by hyperbolicity of the action that if $L\gg 1$ is large, then for the set

$$\{(x,\xi)\in T^*\mathcal{M}\,|\,|\xi|>L\}\cap\bigcup_{k\geq 0}\bigcup_{A\in k\mathcal{O}}e^{\pm X_A^H}(\mathrm{WF}_h(Q')\cap(E_u^*\oplus E_s^*))$$

is contained in $\{|\xi|>L\}\cap\mathcal{C}^L_\pm$ where $\mathcal{C}^L_+$ is a small conic neighborhood of $E_u^*$ and $\mathcal{C}^L_-$ is a small conic neighborhood of $E_s^*$, with the size of the cone sections going to 0 as $L\to+\infty$. Since $T_{\psi_\sigma}(\lambda,s)=R_{\psi_\sigma}(\lambda)(R_{\psi_\sigma}(\lambda)-s)^{-1}$ is independent of $h$, we have $\mathrm{WF}(\mathcal{K}_{T_{\psi_\sigma}(s)})=\mathrm{WF}_h(\mathcal{K}_{T_{\psi_\sigma}(s)})\cap T^*(\mathcal{M}\times\mathcal{M})\setminus\{0\}$, thus by taking $L\to\infty$ we obtain the desired statement, and this holds uniformly in $\theta\in K$. In particular, since $\Omega_+(\mathcal{O})\cup(E_u^*\times E_s^*)$ is disjoint from $N^*\Delta$, we obtain the desired result by choosing a cone $\Gamma$ containing $\Omega_+(\mathcal{O})\cup(E_u^*\times E_s^*)$. $\square$

By the wavefront estimates from Proposition 3.7 it follows that the flat trace $Z_{f,\psi_\sigma}(s,\lambda)$ is well defined and depends meromorphically on $s,\lambda$: indeed, this is a consequence of the fact that the flat trace $\mathrm{Tr}^\flat$ is a continuous linear form on the space $C^{-\infty}_\Gamma(\mathcal{M}\times\mathcal{M})$ equipped with its natural topology given by [Hör03, Definition 8.2.2], provided $\Gamma\cap N^*\Delta=\emptyset$ (the meromorphicity in $\lambda$ can be seen by using the Cauchy characterization of holomorphic functions using contour integrals). To finish the proof of Proposition 3.6, it suffices to prove the expansion (3.8) for some open set of $s,\lambda$. For $\psi\in C^\infty_c(\mathcal{W},\mathbb{R}^+)$ with support near a given $A_0\in\mathcal{W}$ as above and $|\sigma|<\epsilon$ small, since $R_{\psi_\sigma}(\lambda)$ is bounded on $\mathcal{H}^{NG}$, for $|s|\gg 1$ we have as a converging series

$$R_{\psi_\sigma}(\lambda)(s-R_{\psi_\sigma}(\lambda))^{-1}=\sum_{k=1}^\infty s^{-k}R_{\psi_\sigma}(\lambda)^k. \tag{3.20}$$

Formally, if we take the flat trace of the above identity multilied by $f$ on both sides, we obtain using (3.6) the desired identity (3.8).

**Lemma 3.9.** *Let $\psi\in C^\infty_c(\mathcal{W},\mathbb{R}^+)$, $|\sigma|<\epsilon$ and $\psi_\sigma=\psi(\cdot-\sigma)$ as above. Then for each $\lambda\in\mathfrak{a}^*_\mathbb{C}$, if $|s|\gg 1$ is large enough, we obtain (3.8), which we recall:*

$$Z_{f,\psi_\sigma}(s,\lambda)=\sum_{k=1}^\infty s^{-k}\sum_{T\in\mathcal{T}}\sum_{A\in\mathcal{W}\cap L(T)}\frac{\int_T f}{|\det(1-\mathcal{P}_A)|}e^{-\lambda(A)}\psi_\sigma^{(k)}(A).$$

*Proof.* First, as in the rank 1 case, we need an exponential estimate on the number of periodic orbits in the region $\{A\in\mathcal{W}\,|\,|A|\leq L\}$ as $L\to\infty$, which will insure the convergence of the RHS of (3.8) when $|s|\gg 1$ is large. This is the content of Lemma 3.5.

We then follow the argument of [DZ16] in the rank 1 case. By [DZ16, Section 2.4], there is a family of operators $E_\epsilon$ with smooth integral kernels $E_\epsilon(x,y) = C_\epsilon(x)F(d_g(x,y)/\epsilon)$ approximating the Identity as a bounded operator $H^{\epsilon_0}(\mathcal{M}) \to L^2(\mathcal{M})$ as $\epsilon \to 0$ for any fixed small $\epsilon_0 > 0$, with $C_\epsilon(x) = \mathcal{O}(\epsilon^{-n})$ where $n := \dim \mathcal{M}$. Furthermore for each $A : C^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M})$ with $\mathrm{WF}(A) \cap N^*\Delta = \emptyset$ (see [DZ16, Lemma 2.8])

$$\lim_{\epsilon\to 0} \mathrm{Tr}(E_\epsilon A E_\epsilon) = \mathrm{Tr}^\flat(A). \tag{3.21}$$

Moreover, the proof of [DZ16, Lemma 4.1] yields that there is $C > 0$ such that for each $A \in \mathcal{W}$ (with $\|\cdot\|_{\mathrm{Tr}}$ the trace norm) $\|E_\epsilon e^{-X_A} E_\epsilon\|_{\mathrm{Tr}} \le C e^{C|A|}\epsilon^{-n-2}$. Since $\mathrm{supp}(\psi_\sigma^{(k)}) \subset \{A \in \mathfrak{a}\,|\,|A - kA_0| < \delta k\}$ for some small $\delta > 0$ (depending on $\mathrm{supp}(\psi)$) and $A_0 \in \mathcal{W}$ fixed, this implies that there is $C > 0$ such that for all $\epsilon > 0$ and $k \in \mathbb{N}$

$$\|E_\epsilon R^k_{\psi_\sigma}(\lambda) E_\epsilon\|_{\mathrm{Tr}} \le C e^{C\langle\lambda\rangle k}\epsilon^{-n-2}. \tag{3.22}$$

This proof also gives for some uniform $C > 0$

$$\begin{aligned}
|\mathrm{Tr}(E_\epsilon R^k_{\psi_\sigma}(\lambda) E_\epsilon)| &\le \int_{\mathcal{W}} e^{-\mathrm{Re}\lambda(A)} \psi_\sigma^{(k)}(A) |\mathrm{Tr}(E_\epsilon e^{-X_A} E_\epsilon)| dA \\
&\le \int_{\mathcal{W}} \psi_\sigma^{(k)}(A) e^{-\mathrm{Re}\,\lambda(A)} \int_{\mathcal{M}\times\mathcal{M}} E_\epsilon(x,y) E_\epsilon(\varphi^A_{-1}(y), x) dv_g(x) dv_g(y) dA \\
&\le C\epsilon^{-2n} \int_{\mathcal{W}\times\mathcal{M}\times\mathcal{M}} \psi_\sigma^{(k)}(A) e^{-\mathrm{Re}\,\lambda(A)} \mathbf{1}_{\{d_g(x,y)<c_1\epsilon, d_g(x,\varphi^A_{-1}(y))<c_1\epsilon\}} dv_g(x) dv_g(y) dA \\
&\le C\epsilon^{-n} \int_{\mathcal{W}\times\mathcal{M}} \psi_\sigma^{(k)}(A) e^{-\mathrm{Re}\,\lambda(A)} \mathbf{1}_{\{d_g(y,\varphi^A_{-1}(y))<2c_1\epsilon\}} dv_g(y) dA.
\end{aligned}$$

Using Lemma 3.5 and the support property of $\psi_\sigma^{(k)}$, this gives

$$|\mathrm{Tr}(E_\epsilon R^k_{\psi_\sigma}(\lambda) E_\epsilon)| \le C e^{C\langle\lambda\rangle k}. \tag{3.23}$$

Using respectively (3.20), (3.22) and (3.23) we can thus write for $\lambda$ fixed and $|s|$ large enough

$$\begin{aligned}
Z_{f,\psi_\sigma}(s,\lambda) &= \lim_{\epsilon\to 0} \mathrm{Tr}\Big(\sum_{k=1}^\infty s^{-k} E_\epsilon R_{\psi_\sigma}(\lambda)^k E_\epsilon\Big) = \lim_{\epsilon\to 0} \sum_{k=1}^\infty s^{-k} \mathrm{Tr}(E_\epsilon R^k_{\psi_\sigma}(\lambda) E_\epsilon) \\
&= \sum_{k=1}^\infty s^{-k} \lim_{\epsilon\to 0} \mathrm{Tr}(E_\epsilon R^k_{\psi_\sigma}(\lambda) E_\epsilon) = \sum_{k=1}^\infty s^{-k} \mathrm{Tr}^\flat(R^k_{\psi_\sigma}(\lambda))
\end{aligned}$$

where we used (3.21) for the last identity. The formula (3.6) concludes the proof. □

3.3. **Proof of Theorem 4.** Since our analysis is based on the use of the operators $R_\psi(\lambda)$, which are a kind of Laplace transform, it will be convenient to introduce the notation

$$\hat{\psi}(\lambda) := \int_{\mathfrak{a}} e^{\lambda(A)} \psi(A) dA, \ \ \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

(formally, $R_\psi(\lambda) = \hat{\psi}(X + \lambda)$). Using the analytic continuation of the function $Z_{f,\psi}$, we can obtain the following result

**Proposition 3.10.** *Let $\psi \in C_c^\infty(\mathcal{W};\mathbb{R}^+)$ with $\int_{\mathcal{W}} \psi = 1$. For real $\lambda \in \mathfrak{a}^* \subset \mathfrak{a}^*_{\mathbb{C}}$, we denote $\delta = \log \hat{\psi}(\lambda)$. We have for some $\epsilon > 0$*

$$\sum_{T\in\mathcal{T}} \sum_{A\in\mathcal{W}\cap L(T)} \psi^{(k)}(A) e^{-\lambda(A)} \frac{\int_T f}{|\det(1-\mathcal{P}_A)|} = \mu(f)e^{\delta k} + \mathcal{O}(e^{(\delta-\epsilon)k})$$

*Additionally, if $\psi = \psi_\sigma$, the bound on the remainder is locally uniform in $\sigma$.*

*Proof.* The terms we want to estimate are the coefficients $c_k$ in the expansion

$$Z_{f,\psi}(\lambda, s) = \sum_{k\geq 0} s^{-k} c_k.$$

Since $Z_{f,\psi}$ is meromorphic in the $s$ variable in $\mathbb{C}^*$, according to Cauchy's formula, we have for every $\rho \in \mathbb{R}^+$ such that $Z_{f,\psi}(\lambda,\cdot)$ has no poles on the circle of radius $\rho$

$$c_k = \frac{1}{2i\pi}\int_{\rho\mathbb{S}^1} Z_{f,\psi}(\lambda,s)s^{k-1}ds + \sum_{\text{poles of modulus } >\rho} \mathrm{Res}(Z_{f,\psi}(\lambda,s)s^{k-1}ds).$$

Let us assume that $Z_{f,\psi}(\lambda,\cdot)$ has a simple pole $s_0$ with modulus strictly larger than all the other poles, and $s_0$ is real. We denote $K$ the residue at $s_0$, and we find for some $\epsilon > 0$

$$c_k = K s_0^{k-1} + \mathcal{O}((e^{-\epsilon}s_0)^{k-1}).$$

For this reason, we will be done if we can prove that $e^\delta$ is indeed a real dominating pole of order 1 (when $\lambda$ is real), with residue

$$K = \mu(f)e^\delta.$$

Let us describe the poles of $Z_{f,\psi}(\lambda,\cdot)$. As we have seen before, they are exactly the poles of the resolvent $(s - R_\psi(\lambda))^{-1}$. Let us investigate the structure of these poles. Let $s_0 \in \mathbb{C}^*$ be a pole of $s \mapsto (s - R_\psi(\lambda))^{-1}$, and denote by $\Pi(\lambda, s_0)$ the corresponding spectral projector, so that near $s_0$,

$$(s - R_\psi(\lambda))^{-1} = \sum_{j\geq 0} \frac{(R_\psi(\lambda) - s_0)^j \Pi(\lambda, s_0)}{(s-s_0)^{j+1}} + \text{ holomorphic }.$$

Since $s_0 - R_\psi(\lambda)$ is Fredholm on some suitable anisotropic space, the characteristic space $E(s_0)$ is finite dimensional, and since $R_\psi(\lambda)$ commutes with the Anosov action, we can split $E(s_0)$ into a sum of characteristic spaces for the action. We can thus find in $E(s_0)$ a non-zero vector $u$ such that $-X_A u = \lambda_0(A)u$ for some $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ and all $A \in \mathcal{W}$. Since the characteristic space is contained in some suitable anisotropic space, we deduce that $\lambda_0 \in \mathrm{Res}(X)$ where $\mathrm{Res}(X)$ is the set of Ruelle-Taylor resonance of [GBGHW20] (and introduced in (1.19)). It follows that

$$R_\psi(\lambda)u = \int_{\mathcal{W}} e^{-X_A - \lambda(A)}\psi(A)u\,dA = \hat{\psi}(\lambda - \lambda_0)u,$$

so that $s_0 = \hat{\psi}(\lambda-\lambda_0)$. The converse argument completes the proof of the fact that $\mathrm{ran}\,\Pi(\lambda, s_0)$ is exactly equal to

$$\left\{ u \;\middle|\; \lambda_0 \in \mathrm{Res}(X),\ \hat{\psi}(\lambda-\lambda_0) = s_0, \quad u \text{ is a generalized resonant state at } \lambda_0 \right\}.$$

Now, we know from [GBGHW20, Theorem 2] that for all resonances $\lambda_0$, and all $A \in \mathcal{W}$, $\mathrm{Re}(\lambda_0(A)) \leq 0$, with equality for $\lambda_0 = 0$ (if the action is not mixing, there may be other purely imaginary resonances). Now, since we have chosen $\psi$ to be real non-negative, we have

$$\left| \int_{\mathcal{W}} e^{\lambda_0(A)} \psi(A) dA \right| \leq \int_{\mathcal{W}} e^{\mathrm{Re}(\lambda_0(A))} \psi(A) dt.$$

With equality if and only if $e^{\lambda_0(A) - \mathrm{Re}(\lambda_0(A))}$ is constant on the support of $\psi$. It follows that for $\lambda$ real, and $\lambda_0 \in \mathrm{Res}(X)$ with $\lambda_0 \neq 0$,

$$\left| \hat{\psi}(\lambda - \lambda_0) \right| = \left| \int_{\mathcal{W}} e^{(\lambda_0 - \lambda)(A)} \psi(A) dA \right| < \int_{\mathcal{W}} e^{-\lambda(A)} \psi(A) dA = \hat{\psi}(\lambda).$$

This implies that $e^\delta = \hat{\psi}(\lambda)$ is indeed a dominating pole. It remains to compute its order and residue. However from [GBGHW20, Proposition 5.4], we know that there are no Jordan blocks at 0, so that the 0-characteristic space is equal to the 0-eigenspace, and that the corresponding spectral projector $\Pi(\lambda, e^\delta)$ does not depend on the choices – it was denoted $\Pi(0)$ in [GBGHW20]. We deduce that indeed $e^\delta$ is a simple pole, with residue

$$\mathrm{Tr}\, f R_\psi(\lambda) \Pi(0) = e^\delta \mu(f). \qquad \square$$

As a corollory of the proof, we find

**Corollary 3.11.** *The function $Z_{1,\psi}(\lambda, \cdot)$ has simple poles, and*

$$\mathrm{Res}(Z_{1,\psi}(\lambda, s) ds, s_0) = s_0 \sum_{\zeta \in \mathrm{Res}(X), \hat{\psi}(\lambda - \zeta) = s_0} \dim\{u \in C^{-\infty}_{E^u_*} \mid \exists \ell > 0,\ (-X - \zeta)^\ell u = 0\ \}.$$

*Proof.* Near a pole $s_0$ of $(s - R_\psi(\lambda))^{-1}$, we come back to the formula

$$R_\psi(\lambda)(s - R_\psi(\lambda))^{-1} = \sum_{j \geq 0} \frac{R_\psi(\lambda)(R_\psi(\lambda) - s_0)^j \Pi_{s_0}}{(s - s_0)^{j+1}} + \text{ holomorphic around } s_0.$$

Since the trace of nilpotent operators is always 0, taking the trace eliminates the terms $j > 0$, and we deduce that around $s_0$,

$$Z_{1,\psi}(\lambda, s) = \frac{s_0}{s - s_0} \mathrm{Tr}\, \Pi_{s_0} + \text{ holomorphic around } s_0.$$

We have already seen that the range of $\Pi_{s_0}$ is the direct sum of the spaces

$$\{u \in C^{-\infty}_{E^u_*} \mid \exists \ell > 0,\ (-X - \zeta)^\ell u = 0\ \},$$

for $\hat{\psi}(\lambda - \zeta) = s_0$. $\square$

In the case that $f \neq 1$, the singularities could become more complicated because $f$ could interact with the Jordan blocks of $R$. We can now turn to the proof of Theorem 4:

*Proof of Theorem 4.* If suffices to assume $f \geq 0$. We will use some estimates from the proof of [GBGHW20, Proposition 5.4]. Let $\nu \in \mathcal{D}'(\mathcal{W})$ be the measure[9]

$$\nu = \sum_{T \in \mathcal{T}} \sum_{A \in \mathcal{W} \cap L(T)} \frac{\int_T f}{|\det(1 - \mathcal{P}_A)|} \delta_A$$

[9] $\mathcal{D}'(\mathcal{W})$ denotes the space of distributions on the open cone $\mathcal{C}$.

where $\delta_A$ is the Dirac mass at $A$. Choose a basis $(A_j)_{j=1}^{\kappa}$ of $\mathfrak{a}$ so that $A_j \in \ker e_1$ for $j \geq 2$, we then identify $\mathfrak{a} \simeq \mathbb{R}^\kappa$ by identifying the canonical basis of $\mathbb{R}^\kappa$ with the basis $(A_j)_j$. We let $\Sigma = \mathcal{C} \cap \{A_1 + \sum_{j=2}^{\kappa} t_j A_j, \,|\, t_j \in \mathbb{R}\}$ be a hyperplane section of the cone $\mathcal{C}$. Choose $r > 0$ but smaller than the distance of $\Sigma$ to the boundary of the Weyl chamber $\delta W$ (where the distance is the euclidean distance in the chosen coordinates. Next choose $\psi \in C_c^\infty((-r/2, r/2))$ to be non-negative and even with $\int_{\mathbb{R}} \psi = 1$, and for each $\sigma \in \mathbb{R}^\kappa$, define $\psi_\sigma(t) := \prod_{j=1}^{\kappa} \psi(t_j - \sigma_j)$. We view $\Sigma$ as an open subset of $\{(1, \bar{t}) \,|\, \bar{t} \in \mathbb{R}^{\kappa-1}\}$ and choose $q \in C_c^\infty(\Sigma; \mathbb{R}^+)$ with small support and let $Q = \int_{\mathbb{R}^{\kappa-1}} q(\bar{t}) d\bar{t} > 0$, and $\omega \in C_c^\infty((0,1); [0,1])$ with $W := \int_0^1 \omega > 0$. We consider $\lim_{N\to\infty} \nu(F_N)$ for $\sigma(\theta) := (1, \theta) \in \Sigma$ where

$$F_N(t) := \frac{1}{N} \sum_{k=1}^{N} \int_{\mathbb{R}^{\kappa-1}} \omega(\tfrac{k}{N}) \psi_{\sigma(\theta)}^{(k)}(t) q(\theta) d\theta.$$

By applying Proposition 3.10 with $\lambda = 0$, we have

$$WQ\mu(f) = \lim_{N\to\infty} \nu(F_N). \tag{3.24}$$

In the proof of [GBGHW20, Proposition 5.4], it is shown that if $h(t) := t_1^{1-\kappa} \omega(t_1) q(\bar{t}/t_1)$ with $t = (t_1, \bar{t})$, then with $G_N(t) := N^\kappa F_N(tN)$

$$\|G_N(t) - h(t)\|_{L^2(\mathbb{R}^\kappa, dt)} \to 0 \text{ as } N \to \infty.$$

The proof of this convergence is based on the following estimates on the Fourier transform $\hat{G}_N$ of $G_N$: for $\delta > 0$ small, and all $\ell \in \mathbb{N}$ there is $C_\ell > 0, C > 0, c_0 > 0$ so that

$$\forall \xi, |\xi| > N^{1/2+\delta}, \quad |\hat{G}_N(\xi)|^2 \leq C(1 + c_0|\xi|^2)^{-2\epsilon N},$$
$$\forall \xi, |\xi| \leq N^{1/2+\delta}, \quad |\hat{G}_N(\xi)|^2 \leq C_\ell(|\xi|^{-\ell} + N^{-\ell(\frac{1}{2}-\delta)})$$

and $G_N(\xi) \to \hat{h}(\xi)$ for each $\xi \in \mathbb{R}^\kappa$. As a consequence, for each $\ell > 0$ one can take $N_0$ large enough so that for all $N > N_0$

$$\int_{|\xi| > N^{1/2+\delta}} \langle \xi \rangle^{2\ell} |\hat{G}_N(\xi) - \hat{h}(\xi)|^2 d\xi \leq C_\ell \int_{|\xi| > N^{1/2+\delta}} \langle \xi \rangle^{-\ell} d\xi = o(1),$$

$$\mathbf{1}_{|\xi| \leq N^{1/2+\delta}} \langle \xi \rangle^{2\ell} |\hat{G}_N(\xi) - \hat{h}(\xi)|^2 \leq C_\ell \langle \xi \rangle^{-\ell},$$

where $C_\ell > 0$ is independent of $N$. Using Lebesgue dominated convergence theorem, we conclude that $\|\langle \xi \rangle^\ell (\hat{G}_N - \hat{h}(\xi))\|_{L^2} \to 0$ as $N \to \infty$, thus by Sobolev embedding $\|G_N - h\|_{C^0} \to 0$ as $N \to 0$. Therefore

$$\|F_N(t) - N^{-\kappa} h(t/N)\|_{C^0} = o(N^{-\kappa}). \tag{3.25}$$

Choosing $q = 1$ on an open set $U \subset \Sigma$ and $\omega(t_1) = t_1^{\kappa-1}$ in $(\epsilon, 1-\epsilon)$, we deduce that for $N$ large enough

$$F_N(t) \geq \frac{1}{2N^\kappa} \mathbf{1}_{[\epsilon N, N(1-\epsilon)]}(t_1) \mathbf{1}_U(\bar{t}/t_1).$$

This implies that if $\mathcal{C}_{a,b}(U) := \{(t_1, \bar{t}) \,|\, t_1 \in [a,b], \bar{t}/t_1 \in U\}$

$$\nu(\mathcal{C}_{\epsilon N, N(1-\epsilon)}(U)) \leq 2N^\kappa \nu(F_N) \leq 3N^\kappa QW\mu(f)$$

where we used (3.24). Thus we obtain $\nu(\mathcal{C}_{0,N}(U)) = \mathcal{O}(N^\kappa)$ by letting $\epsilon \to 0$. This estimate thus also implies by a covering argument that $\nu(\mathcal{C} \cap \{|A| < N\}) = \mathcal{O}(N^\kappa)$. Coming back to a general $\omega, q$, using that $\mathrm{supp}(F_N) \cup \mathrm{supp}(h(\cdot/N)) \subset \mathcal{C}_{0,3N}(U')$ for some open set $U' \in \mathcal{W} \cap \{t_1 = 1\}$, and since $\nu(\mathcal{C}_{0,3N}(U')) = \mathcal{O}(N^\kappa)$, we obtain by (3.25)

$$\lim_{N\to\infty} N^{-\kappa}\nu(h(\cdot/N)) = \lim_{N\to\infty} \nu(F_N) = WQ\mu(f).$$

Next, let $U$ be a small open ball in $\Sigma$, we can choose $q_j \in C_c^\infty(\Sigma)$ supported near $U$ for $j = 1, 2$ so that $q_1 \le \mathbf{1}_U \le q_2$ and $\int q_j = |U| + \mathcal{O}(\epsilon)$, and for $0 < a < b < 1$, choose $\omega_j \in C_c^\infty((0,1),[0,1])$ such that $\omega_1 \le t^{\kappa-1}\mathbf{1}_{[a,b]} \le \omega_2$ with $\int_0^1 \omega_j(t)dt = \int_a^b t^{\kappa-1}dt + \mathcal{O}(\epsilon)$. Write now $h_j(t) = t_1^{1-\kappa}\omega_j(t_1)q_j(\bar{t}/t_1)$. One then has

$$N^{-\kappa}\nu(h_1(\cdot/N)) \le N^{-\kappa}\nu(\mathcal{C}_{\delta N,(1-\delta)N}(U)) \le N^{-\kappa}\nu(h_2(\cdot/N))$$

thus if $V_{a,b} = \int_a^b t^{\kappa-1}dt$, we obtain for each $\epsilon > 0$ small

$$(V_{a,b} - \epsilon)(|U| - \epsilon)\mu(f) \le \liminf_{N\to\infty} N^{-\kappa}\nu(\mathcal{C}_{aN,bN}(U))$$
$$\limsup_{N\to\infty} N^{-\kappa}\nu(\mathcal{C}_{aN,bN}(U)) \le (V_{a,b} + \epsilon)(|U| + \epsilon)\mu(f).$$

We let $\epsilon \to 0$ and deduce that

$$\lim_{N\to\infty} N^{-\kappa}\nu(\mathcal{C}_{aN,bN}(U)) = \mu(f)V_{a,b}|U| = \mu(f)|\mathcal{C}_{a,b}(U)| \qquad \square.$$

3.4. **A dynamical zeta function.** We conclude by some comments on links with dynamical zeta functions. In [BT08], for an operator $K$, Baladi and Tsujii introduce the flat determinant as

$$\det{}^\flat(1 - zK) := \exp\left(-\sum_{\ell\ge 0} \frac{z^\ell}{\ell} \operatorname{Tr}^\flat K^\ell\right),$$

provided the LHS makes sense and converges. From the argument of the previous section, we see that $\det^\flat(1 - zR_\psi(\lambda))$ is well defined for $|z|$ small enough, and has a holomorphic extension to $z \in \mathbb{C}$. Indeed for $|z|$ small enough,

$$\frac{\partial_z \det^\flat(1 - zR_\psi(\lambda))}{\det^\flat(1 - zR_\psi(\lambda))} = -\frac{1}{z}\sum_{\ell\ge 1} z^\ell \operatorname{Tr}^\flat R_\psi(\lambda)^\ell = -\frac{1}{z} Z_{1,\psi}\left(\lambda, \frac{1}{z}\right).$$

As the RHS is meromorphic on $\mathbb{C}$ in the $z$ variable, it remains to show it has simple poles, with integer residue in order to deduce that $\det^\flat(1 - zR_\psi(\lambda))$ is holomorphic. According to Corollary 3.11, near a pole $1/s_0$, $Z_{1,\psi}$ takes the form

$$-\frac{1}{z} Z_{1,\psi}\left(\lambda, \frac{1}{z}\right) = -\frac{s_0 m}{z(1/z - s_0)} + \text{ holomorphic} = \frac{m}{z - 1/s_0} + \text{ holomorphic}.$$

Here $m$ is the dimension of some characteristic space, i.e an integer. This proves that $\lambda, z \mapsto \det^\flat(1 - zR_\psi(\lambda))$ is holomorphic in $\mathfrak{a}^*_{\mathbb{C}} \times \mathbb{C}$. The parameter $z$ here is auxiliary, so we can fix its value to 1, and obtain:

**Theorem 5.** *Let $X$ be an Anosov action of $\mathbb{R}^\kappa$ with positive Weyl chamber $\mathcal{W}$, and let $\psi \in C_c^\infty(\mathcal{W}, \mathbb{R}^+)$ have $\int \psi = 1$ and small enough support. Then*

$$d_\psi(\lambda) := \exp\left(-\sum_{T\in\mathcal{T}} \sum_{A\in L(T)} \frac{\mathrm{vol}(T)}{|\det(1 - \mathcal{P}_A)|} e^{-\lambda(A)} \sum_k \frac{\psi^{(k)}(A)}{k}\right),$$

*originally defined for* $\operatorname{Re}\lambda$ *large enough (in the sense of evaluating it in elements of the positive Weyl chamber* $\mathcal{W}$*), has a holomorphic continuation to* $\mathfrak{a}^*_{\mathbb{C}}$*. This continuation is a regularized version of the formal product*

$$\prod_{\zeta\in\operatorname{Res}(-X)}(1-\hat{\psi}(\lambda-\zeta))^{\text{multiplicity of }\zeta}.$$

*This means that for each* $\lambda_0\in\mathbb{C}^\kappa$*, there exist only a finite number of* $\zeta$*'s in* $\operatorname{Res}(X)$ *such that* $|\hat{\psi}(\lambda_0-\zeta)-1|<1/2$*, and for* $\lambda$ *close enough to* $\lambda_0$*,*

$$d_\psi(\lambda)=\prod_{\zeta\in\operatorname{Res}(X)}(1-\hat{\psi}(\lambda-\zeta))^{\text{multiplicity of }\zeta}\times\text{ a holomorphic function.}$$

As far as we know, this is the first appearance of a multi-parameter zeta function for actions with a global meromorphic extension. The reader accustomed to the rank 1 case may find the formula a bit surprising and wonder if it is possible to replace the term $F:=\sum\psi^{(k)}/k$ by a simpler weight function. Let us explain briefly why this is not an easy question. The natural extension of the rank 1 case would be to replace it by

$$\tilde{F}:=\mathbf{1}_{\{\lambda_j>0,\ j=1\dots\kappa\}},$$

having taken a basis comprised of elements of $\mathcal{W}$ and $\lambda=(\lambda_1,\dots,\lambda_\kappa)$ in this basis. However, there seems to be no hope that this can define a globally meromorphic function. Indeed, let us consider the singular set of such a function. Each resonance $\zeta$ would contribute by

$$\left(\prod\frac{1}{\lambda_j-\zeta_j}\right)^{\text{multiplicity of }\zeta}.$$

For the resulting product to be meromorphic, the singular set has to be (at least) locally closed. It is given by

$$\{\lambda\in\mathbb{C}^\kappa\mid\exists\zeta\in\operatorname{Res}(X),\ \exists j=1\dots\kappa,\lambda_j=\zeta_j\}.$$

For this to be locally closed, we need that for any sequence of resonances $\zeta^\ell$, each coordinate $(\zeta_j^\ell)_\ell$ tends to infinity. For example for the resonances with small real part, this means that the imaginary parts cannot equidistribute in $\mathbb{R}^\kappa$. This would certainly be a surprise to us, in particular as for Weyl chamber flows a Weyl-lower bound on the number of Ruelle Taylor resonances with $\operatorname{Re}(\lambda)=-\rho$ is known [HWW21, Theorem 1.1].

## 4. Application to lattice point counting

In this final section we will work out the consequences of the Bowen formula for the SRB measures in terms of the counting-problem of lattice points. We focus in particular on the case of Weyl chamber flows, where we obtain precise estimates for the exponential growth rates.

Choose a proper subcone $\mathcal{C}\subset\mathcal{W}$ and fix some $\xi\in\mathfrak{a}^*$ such that $\xi$ is positive on a small conical neighbourhood of $\mathbb{C}$. For $0\le a<b$ we define $\mathcal{C}_{a,b}:=\{A\in\mathcal{C},\xi(A)\in[a,b]\}$ and the lattice point counting function

$$\mathcal{N}_{\mathcal{C}_{a,b}}:=\sum_{T\in\mathcal{T}}\sum_{A\in L(T)\cap\mathcal{C}_{a,b}}\operatorname{vol}(T).$$

We furthermore introduce a counting function that is additionally weighted by the Jacobians $|\det(1-\mathcal{P}_A)|$

$$\mathcal{N}^w_{\mathcal{C}_{a,b}} := \sum_{T\in\mathcal{T}} \sum_{A\in L(T)\cap\mathcal{C}_{a,b}} \frac{\mathrm{vol}(T)}{|\det(1-\mathcal{P}_A)|}.$$

Note that by Theorem 4 choosing the constant test function $f=1$ we get for any $q>1$

$$\lim_{n\to\infty} \frac{\mathcal{N}^w_{\mathcal{C}_{q^{n-1},q^n}}}{q^{\kappa(n-1)}} = |\mathcal{C}_{1,q}| \tag{4.1}$$

Let us define

$$m := \liminf_{n\to\infty} \frac{\log\left(\inf_{A\in\mathcal{C}_{q^{n-1},q^n}\cap(\cup_{T\in\mathcal{T}}L(T))}|\det(1-\mathcal{P}_A)|\right)}{q^n}$$

$$M := \limsup_{n\to\infty} \frac{\log\left(\sup_{A\in\mathcal{C}_{q^{n-1},q^n}\cap(\cup_{T\in\mathcal{T}}L(T))}|\det(1-\mathcal{P}_A)|\right)}{q^n}$$

Then for any $\varepsilon>0$ there is $N$ such that for $n>N$ we have

$$e^{(m-\varepsilon)q^n}\mathcal{N}^w_{\mathcal{C}_{q^{n-1},q^n}} \le \mathcal{N}_{\mathcal{C}_{q^{n-1},q^n}} \le \mathcal{N}^w_{\mathcal{C}_{q^{n-1},q^n}} e^{(M+\varepsilon)q^n}$$

and taking additionally (4.1) into account we get

$$(|\mathcal{C}_{1,q}|-\varepsilon)q^{\kappa(n-1)}e^{(m-\varepsilon)q^n} \le \mathcal{N}_{\mathcal{C}_{q^{n-1},q^n}} \le (|\mathcal{C}_{1,q}|+\varepsilon)q^{\kappa(n-1)}e^{(M+\varepsilon)q^n}.$$

Now, using $\mathcal{N}_{\mathcal{C}_{0,q^n}} = \mathcal{N}_{\mathcal{C}_{0,1}} + \sum_{k=0}^n \mathcal{N}_{\mathcal{C}_{q^{k-1},q^k}}$ we deduce

$$m \le \liminf_{n\to\infty} \frac{\log\mathcal{N}_{\mathcal{C}_{0,q^n}}}{q^n} \le \limsup_{n\to\infty} \frac{\log\mathcal{N}_{\mathcal{C}_{0,q^n}}}{q^n} \le M$$

Now let us assume that there is $\eta\in\mathfrak{a}^*$ positive on $\mathcal{W}$ such that for any proper subcone $\mathcal{C}\subset\mathcal{W}$ there is $\varepsilon>0$ such that $|\det(1-\mathcal{P}_A)| = e^{\eta}(A)(1-\mathcal{O}(e^{-\varepsilon|A|}))|$ for all $A\in\mathbb{C}$. If we now fix $\xi := \eta/\|\eta\|$ then we get the particularly simple expressions $M=\|\eta\|$ and $m=\|\eta\|/q$. As we can choose $q>1$ arbitrary close to 1 we get:

**Proposition 4.1.** *For an Anosov action for which there is $\eta\in\mathfrak{a}^*$ with the above properties, one has for each proper subcone $\mathcal{C}\subset\mathcal{W}$*

$$\lim_{R\to\infty} \frac{\log\mathcal{N}_{\mathcal{C}_{0,R}}}{R} = \|\eta\|$$

.

The assumption on $|\det(1-\mathcal{P}_A)|$ is fulfilled for all standard Anosov actions. For example for Weyl chamber flows, a periodic point $x_0 = \Gamma g_0 M \in \Gamma\backslash G/M = \mathcal{M}$ under $A_0\in\mathcal{W}$ implies the existence $\gamma_0\in\Gamma, m_0\in M$ such that $\gamma_0 g_0\exp(A_0)m_0 = g_0$. With these notation we can give an explicit expression of $|\det(1-\mathcal{P}_{A_0})|$

$$\begin{aligned}|\det(1-\mathcal{P}_{A_0})| &= \prod_{\alpha\in\Delta_+}\left|\det_{\mathfrak{g}_\alpha}\left(1-e^{-\alpha(A_0)}\mathrm{Ad}(m_0^{-1})\right)\right|\left|\det_{\mathfrak{g}_{-\alpha}}\left(1-e^{\alpha(A_0)}\mathrm{Ad}(m_0^{-1})\right)\right| \\ &= e^{2\rho(A_0)}\prod_{\alpha\in\Delta_+}\left|\det_{\mathfrak{g}_\alpha}\left(1-e^{-\alpha(A_0)}\mathrm{Ad}(m_0^{-1})\right)\right|\left|\det_{\mathfrak{g}_{-\alpha}}\left(\mathrm{Ad}(m_0^{-1})-e^{-\alpha(A_0)}\right)\right|\end{aligned}$$

Here $\Delta_+ \subset \mathfrak{a}^*$ denotes the set of positive roots, $\mathfrak{g}_{\pm\alpha}$ the corresponding root spaces, $m_\alpha := \dim \mathfrak{g}_\alpha$ and we use the usual notation $\rho := \sum_{\alpha\in\Delta_+} \frac{m_\alpha}{2}\alpha \in \mathfrak{a}^*$ for the half sum of positive roots. As the adjoint action of $M$ on $\mathfrak{g}_\alpha$ is orthogonal and for any proper subcone $\mathcal{C} \subset \mathcal{W}$, $\alpha(A) > \varepsilon|A|$ for all $A \in \mathcal{C}$ we deduce by the continutiy of the determinant

$$|\det(1 - \mathcal{P}_A)| = e^{2\rho(A)}(1 + \mathcal{O}(e^{-\varepsilon|A|}) \tag{4.2}$$

## Appendix A. Upper bound on the number of periodic orbits (Lemma 3.5)

First, for $\delta_0 > 0$ small there are is a family of invertible linear maps $\mathcal{T}_{x,y} : T_x\mathcal{M} \to T_y\mathcal{M}$ depending continuously on $d_g(x, y) \le \delta_0$ such that $\mathcal{T}_{x,x} = \mathrm{Id}$ and $\mathcal{T}_{x,y}$ mapping $E_u(x), E_s(x)$, and $E_0(x)$ onto $E_u(y), E_s(y)$ and $E_0(y)$. Then exactly the same proof as [DZ16, Lemma A.1] shows that for each $r > 0$ there is $\delta \in (0, \delta_0)$ and $C > 0$ such that if $d_g(x, \varphi_1^A(x)) < \delta$, $A \in \mathcal{W}$ with $|A| > r$ and $v \in E_u(x) \oplus E_s(x)$, then

$$|v| \le C|(d\varphi_1^A - \mathcal{T}_{x,\varphi_1^A(x)})v|. \tag{A.1}$$

This should be compared to the fact that for $A \in \mathcal{W}$, $\varphi_1^A(x) = x$ we know that $(d_x\varphi_1^A - \mathrm{Id})_{|E_u(x)\oplus E_s(x)}$ is invertible. (A.1) generalizes this invertibility to orbits that are only approximately closed.

Next, we have, by the group property of $e^{X_\cdot}$, that there is $C > 0, M > 0$ such that for all $A \in \mathcal{W}$, $x, x' \in \mathcal{M}$

$$\|e^{X_A}\|_{C^2\to C^2} \le Ce^{M|A|}, \quad d_g(\varphi_1^A(x), \varphi_1^A(x')) \le Ce^{M|A|}d_g(x, x'). \tag{A.2}$$

Next, we show a separation estimate between periodic tori.

**Lemma A.1.** *Let $r > 0$, then there is $C, \delta > 0$ such that for all $\epsilon > 0$ small, if $d_g(x, \varphi_1^A(x)) \le \epsilon$, $d_g(x', \varphi_1^{A'}(x')) \le \epsilon$, $A, A' \in \mathcal{W}$ with $|A - A'| \le \delta$, $d_g(x, x') \le \delta e^{-M|A|}$, then $|A - A'| \le C\epsilon$, and furthermore there is $A'' \in \mathfrak{a}$ with $|A''| \le 1$ such that $d_g(x, \varphi_1^{A''}(x')) \le C\epsilon$.*

Letting $\varepsilon \to 0$ we get, as a direct consequence

**Corollary A.2.** *If two periodic orbits $\cup_{t\in[0,1]}\varphi_t^A(x)$ and $\cup_{t\in[0,1]}\varphi_t^{A'}(x')$ have minimal distance $\le \delta e^{-M|A|}$ and nearby period $|A - A'| < \delta$, and $A, A' \in \mathcal{W}$, then $A = A'$ and there is an invariant torus orbit $T$ such that $x, x' \in T$.*

*proof of Lemma A.1.* We follow closely the proof of [DZ16, Lemma A.2]. Under our assumptions, $x, x', \varphi_1^A(x)$ and $\varphi_1^{A'}(x')$ are all in a small chart in $\mathbb{R}^n$ and we will frequently identify points in $\mathcal{M}$ and vectors in $T\mathcal{M}$ via as elements in $\mathbb{R}^n$ via this chart. The norm $|\bullet|$ then induces a metric that is equivalent to the Riemannian distance on $\mathcal{M}$. We will furthermore assume that chart is chosen small enough such that for any $x, x'$ in the chart, the angle of $E_u(x) \oplus E_s(x)$ and $E_0(x')$ is bounded from below. As $E_u(x) \oplus E_s(x)$ is a slice of the $\mathbb{A}$ action, there is $A'' \in \mathfrak{a}$ with $|A''| \le 1$ such that $\varphi_1^{A''}(x') - x \in E_u(x) \oplus E_s(x)$. We write $x'' := \varphi^{A''}(x')$. By the boundedness of the angles between $E_u \oplus E_s$ and $E_0$ there is a global $C$ such that $|x, x''| \le C|x, x'|$. Then by Taylor expansion there is $C > 0$ such that

$$|\varphi_1^A(x'') - \varphi_1^A(x) - d\varphi_1^A(x)(x'' - x)| \le Ce^{M|A|}|x - x''|^2 \le C\delta|x - x''|,$$
$$|\varphi_1^{A'}(x'') - \varphi_1^A(x'') - X_{A'-A}(\varphi_1^A(x''))| \le C|A - A'|^2 \le C\delta|A - A'|$$

thus we obtain

$$|\varphi_1^{A'}(x'') - \varphi_1^A(x) - d\varphi_1^A(x)(x'' - x) - X_{A'-A}(\varphi_1^A(x''))| \le C\delta(|x - x''| + |A - A'|).$$

Then, using $d_g(x,\varphi_1^A(x))\le\epsilon$ and $d_g(x'',\varphi_1^{A'}(x''))\le C\epsilon$ for some uniform $C>0$,

$$|(d\varphi_1^A(x)-\mathrm{Id})(x''-x)+X_{A'-A}(\varphi_1^A(x''))|\le C\delta(|x-x''|+|A'-A|)+C\epsilon.$$

Using that $\mathcal{T}_{x,y}$ is uniformly continuous in $x,y$, and $d_g(x,\varphi_1^A(x))\le\epsilon$ we get, if $\epsilon$ is chosen small enough (depending on $\delta$):

$$|(d\varphi_1^A(x)-\mathcal{T}_{x,\varphi_1^A(x)})(x''-x)+X_{A'-A}(\varphi_1^A(x''))|\le C\delta(|x''-x|+|A'-A|)+C\epsilon.$$

Finally, using that $(d\varphi_1^A(x)-\mathcal{T}_{x,\varphi_1^A(x)})(x''-x)\in(E_u\oplus E_s)(\varphi_1^A(x))$, $X_{A'-A}(\varphi_1^A(x''))\in E_0\varphi_1^A(x'')$ together with the lowe bound on the angle of these subspaces as well as (A.1), we conclude that

$$|A-A'|+|x-x''|\le C(|d\varphi_1^A(x)-\mathcal{T}_{x,\varphi_1^A(x)})(x''-x)|+|A'-A|)\le C\delta(|A'-A|+|x''-x|)+C\epsilon$$

which gives the result by choosing $\delta$ small enough. ☐

We now prove (3.5). Let $\ell>0$ be large. We take a maximal set of points $(x_j,A_j)_j$ in $\mathcal{M}\times\{A\in\mathcal{W}||A|\le\ell\}$ so that $d_g(x_j,x_k)>\delta e^{-M\ell}/2$ or $|A_j-A_k|>\delta/2$. The number of such balls is $\mathcal{O}(\ell^\kappa e^{nM\ell})$ and the polynomial term in $\ell$ can easily be absorbed (by changing $M$) such that we have $\mathcal{O}(e^{nM\ell})$ balls. One has

$$Z:=\{(x,A)\in\mathcal{M}\times\mathcal{W}\,|\,|A|\le\ell,d_g(x,\varphi_1^A(x))\le\epsilon\}\subset\bigcup_j B_j,$$

$$B_j:=\{(x,A)\in\mathcal{M}\times\mathcal{W}\,|\,|A-A_j|\le\delta/2,d_g(x,x_j)\le\delta e^{-ML}/2,d_g(x,\varphi_1^A(x))\le\epsilon\}.$$

Now by Lemma A.1, if $(x',A')\in B_j$, $B_j$ is contained in an $\epsilon$-neighborhood of the orbit $\{\varphi^{A''}(x')\,|\,|A''|\le1\}$ times a $\epsilon$ ball of $A'$ in $\mathcal{W}$. The first neighbourhood has $\nu_g$-measure $\mathcal{O}(\epsilon^{n-\kappa})$ and the latter $dA$-measure $\mathcal{O}(\epsilon^\kappa)$. This shows that $(v_g\otimes dA)(Z)=\mathcal{O}(\epsilon^n e^{nM\ell})$.

We conclude with a bound on the number of periodic tori. By Corollary A.2, we see that the periodic tori $T$ so that $L(T)\cap B(A,\delta)\ne\emptyset$ with $B(A,\delta):=\{A'\in\mathcal{W}\,|\,|A-A'|<\delta\}$ are separated by a distance at least $\delta e^{-M|A|}$, thus there are tubular neighborhoods of volume bounded below by $C\delta^{n-\kappa}e^{-(n-\kappa)M|A|}$ that do no intersect in $\mathcal{M}$. By a covering argument we deduce that for each $A\in L(T)$ with $T\in\mathcal{T}$

$$\sharp\{T'\in\mathcal{T}\,|\,L(T')\cap B(A,\delta)\ne\emptyset\}\le C\delta^{\kappa-n}e^{(n-\kappa)M|A|}$$

$$\sharp\{A'\in L(T')\cap\mathcal{W}\,|\,T'\in\mathcal{T},A'\in B(A,\delta)\}\le C\delta^{\kappa-n}e^{(n-\kappa)M|A|}$$

and therefore again by covering $\mathcal{W}\cap\{|A|\le\ell\}$ by $\mathcal{O}(\ell^\kappa)$ balls of radius $\delta$ we conclude that

$$\sharp\{A\in L(T)\cap\mathcal{W}\,|\,T\in\mathcal{T},|A|\le\ell\}\le C\ell^\kappa e^{(n-\kappa)M\ell}.$$

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*Email address*: bonthonneau@math.univ-paris13.fr

LAGA, Institut Galilée, 99 avenue Jean Baptiste clément, 93430 Villetaneuse, France.

*Email address*: colin.guillarmou@universite-paris-saclay.fr

Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France.

*Email address*: weich@math.upb.de

Universität Paderborn, Warburgerstr. 100, 33098 Paderborn, Germany