High-order embedding formalism, Noether's theorem on time scales and Eringen's nonlocal elastica

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High-order Embedding Formalism, Noether's Theorem on Time Scales and Eringen's Nonlocal Elastica

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Contents

Dedicated to

my parents, *Salah*, and *Mabrouka*

my wife, *Ouissam*, and my son, *Anis*

my brothers, *Hamza*, *Belal*, and *Ahmed Yacine*

my sisters, *Romaissa* and *Maroua*

my sister's kids, *Youcef*, *Sojoud* and *Ritaj*

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General Introduction

The main topic of this Thesis is the **connection between continuous and discrete versions** of a given object, typically an ordinary differential equation. This problem has a long history and we refer to the paper by A. Lesne [67] for a presentation of some historical examples in physics. This connection can be studied in (at least) two different directions: one going from a continuous setting to a discrete analogue, and in a symmetric way, from a discrete setting to a continuous one. The first procedure is typically used in numerical analysis in order to construct numerical integrators and the second one is typical of continuous modeling for the study of micro-structured materials.

We can not hope to obtain a general or global answer for all the questions raised by the continuous versus discrete modeling. We then focus our attention on three distinct problems.

In Part A, we propose a general framework precising different ways to derive a discrete version of a differential equation called **discrete embedding formalisms**.

Part B focuses on the preservation of symmetries for discrete versions of Lagrangian and Hamiltonian systems, i.e. analogue of Noether's theorem.

Finally, Part C applies these results in mechanics, precisely for the derivation of a discrete Eringen's nonlocal elastica.

We now precise our contributions in these three parts.

Part A -Discrete embedding formalisms and high-order time-scale calculus

In order to discuss these two processes, we introduce first a general framework called "**Embedding formalisms**" first initiated in [16], [24], [25] and developed in a discrete setting in [26], [29], [30].

The discussion and construction of embedding formalisms is the subject of the Part A of the Thesis.

What is an embedding formalism ?

Informally, this is a combination of a general construction used to obtain analogue of the classical derivative, anti-derivative, functionals, etc, over functional sets which are not usual and a set of rules to manipulate these operators. As a consequence, such a formalism can be used to obtain extensions of classical ordinary differential equations or partial differential equations. An embedding depends on the specific structure used to generalize an equation. Indeed, an ordinary differential equation

$$\frac{dx}{dt} = f(x,t), \tag{.0.1}$$

can be encoded by the data of the associated differential operator

$$O := \frac{d}{dt} - f(\cdot,t), \tag{.0.2}$$

its integral form

$$x(t) = x_0 + \int_0^t f(x(s),s)\,ds, \tag{.0.3}$$

or by its variational structure when it exists, i.e. a functional

$$\mathscr{L}(x) = \int_a^b L\left(s, x(s), \frac{dx}{ds}\right)\,ds. \tag{.0.4}$$

This list is of course not exhaustive. All these presentations are equivalent in a classical setting but can lead to non equivalent equations when generalized. In this manuscript, we focus on the **differential, integral and variational embeddings**. Namely, denoting by X an object on the new functional space, D the new derivative and I the new antiderivative, then the differential embedding is given by

$$D[X] = f(X,t), \tag{.0.5}$$

as long as $f(X,t)$ has a precise meaning. The integral embedding is given by

$$X(t) = X_0 + I_0^t\left[f(X(s),s)\right], \tag{.0.6}$$

and the variational embedding is obtained from the functional

$$\mathscr{L}(X) = I_a^b\left[L(s,X(s),D[X](s))\right], \tag{.0.7}$$

by developing a new calculus of variations adapted to this kind of functional.

Despite the fact that previous general overviews of embedding formalisms exist in the literature [24], [26], [29], [30], an abstract presentation in the spirit of category theory is still missing.

We provide a general framework in Chapter II together with F. Pierret from the Observatory of Paris based on the previous work [29], [30].

Embedding formalisms are algebraic in nature and give a precise connection with the classical differential calculus. We can then study how certain properties satisfied by some ODEs are preserved or modified by an embedding. A typical example is given by ODEs possessing symmetries: under which conditions such symmetries are preserved and if not, which consequences can be expected for the behavior of the solutions in the new setting ?

In this Ph.D. we are specially interested in discrete embedding formalisms. We focus also on the preservation of variational structures (Lagrangian, Hamiltonian) and symmetries under embeddings.

A natural framework is then given by **time-scale calculus** defined by S. Hilger [52] in 1988 which already gives an explicit setting allowing to write equations both in a continuous, discrete or mixed setting. Embeddings on time-scales have been studied in [28]. Despite its interest, time-scale calculus is typically at the discrete level an order one theory, meaning that all continuous operators (derivatives or integrals) are approximated up to order one. This property induces some limitations for its application to numerical analysis where high order numerical scheme are looked for.

An important issue is then to extend this formalism to "higher orders".

This is done in Chapter III in collaboration with A. Szafranska from the Gdansk University of Technology.

Having such a framework, we look for the **variational embedding of Lagrangian systems over time-scale**. In the discrete setting, variational embeddings of a given second order differential equation lead to **variational integrators** as defined by J.E. Marsden and M. West in [74] which correspond to a special class of **geometric numerical integrators** as exposed in [44].

A natural question is then:

What is the advantage of considering variational integrators as a consequence of variational embeddings with respect to the classical construction used for example by J.E. Marsden and M. West ?

The main difference is related to the use of discrete operators which are clearly associated to their continuous counterpart. This allows to understand very clearly how the properties of these operators modify the associated calculus of variations and as a consequence of the associated Euler-Lagrange equation. Algebraic properties of these operators play then a fundamental role and explain how far is the new calculus (here discrete or time-scale) from the classical differential calculus. Natural questions are then:

Do we have a fundamental theorem of differential calculus in the discrete or time-scale setting ? What is the analogue of the integration by part formula ? Do we have a chain rule formula ?

Classical results on time-scale calculus [2] already provide some answers but these questions have to be studied in the high-order case.

This is done in Chapter III where the properties of the high-order derivative and integrals are studied.

The construction of *higher order variational integrators* has been developed in a serie of papers by M. Leok [20], [48], [68], [69] and are called *the Galerkin variational integrators*. The analysis of these methods is done by S. Ober-Blöbaum [82], J. Hall [46] and J. Hall and M. Leok [47]. Despite all these works, the formulation of the discrete Euler-Lagrange equation is not transparent and does not have a clear connection with the classical one. A formulation in terms of discrete operators is then needed.

As a consequence, one is lead to the study of the **calculus of variations on time-scales**. This topic has been developed firstly by M. Bohner [10] in 2004, followed by numerous generalizations ([14], [37], [76], [88]). Indeed, as we have different choices for the time-scale derivative and time-scale integrals in order to define the time-scale Lagrangian functional, we have different versions of calculus of variations on time-scales called Δ or ∇ calculus of variations and shifted or non-shifted. We refer to Chapter V for more details.

The calculus of variations on high-order has to be formalized using the high-order discrete operators.

This is done in Chapter III where the high-order discrete Euler-Lagrange equation is derived.

Part B- Time-scale Noether's theorems for Lagrangian and Hamiltonian systems

Symmetries play a crucial role in physics and in mathematics and in particular for the study of differential equations [83]. Lagrangian systems have a special place in this setting due to **Noether's theorem** first stated in 1918 [61]: invariance of the functional by a group of symmetries (variational symmetries) induces *constants of motion*.

A natural question is then: Assume that a discrete (or time-scale) functional is invariant (in a sense to be precised) under a group of symmetries. Can we state an analogue of Noether's theorem ?

This problem has been studied by an large number of people. In the discrete case, previous results have been given by S. Maeda in the 80's [70], [71] followed by similar results in the context of variational integrators by J.E. Marsden's school in particular [74]. We refer to the book [45] for more references.

In the time-scale setting, the classical reference is the article by Z. Bartosiewicz and D.F.M. Torres [8].

This article deals with general group of symmetries in particular, actions acting on the time variable. The case of group of symmetries acting without time is easier to treat and has been derived several times (see [14] for an overview). The technique of proof used by Z. Bartosiewicz and D.F.M. Torres is based on a method exposed by J. Jost in [58] that we call **Jost's method** for simplicity in the following. This method has been used in different contexts to obtain generalizations of Noether's theorem but leads to difficulties (see [32]).

In Chapter V, we prove that the result of Z. Bartosiewicz and D.F.M. Torres in [8] is not correct and as a consequence, all the results supported by this article, in particular the one related to the second-order Euler-Lagrange equation [7], [73]. Moreover, following the same strategy, i.e. the Jost method of proof or the direct method, we provide a time-scale Noether's theorem which corrects [8].

Using the same method of proof than Z. Bartosiewicz and D.F.M. Torres, several authors have tried to **derive a Hamiltonian version of Noether's theorem on time-scale**. We refer in particular to the work of K. Peng and Y. Luo [84], X.H. Zhai and L.Y. Zhang [90] and Song L.Y. Zhang [87] where they make use of the time-scale version of the second-order Euler-Lagrange equation which was proved to be wrong.

In Chapter VI, together with A. Hamdouni and J. Palafox from the University of La Rochelle, we prove a Hamiltonian version of Noether's theorem on time scales for different versions of the time-scale calculus of variations (Δ and ∇ calculus of variations, shift or nonshifted).

Part C- Continuous and discrete Eringen's nonlocal elastica

All the previous materials can be used to study the following problem suggested to us by N. Challamel from Bretagne-Sud University:

Eringen's nonlocal elastica equation is a classical equation of mechanics studied by N. Challamel, Kocsis and Wang [23] obtained by the continualization method.

A natural question is to derive a discrete analogue of this mechanical system. Several possibilities exist and N. Challamel explores some of them in [23].
In this Part, we construct a discrete version of Eringen's nonlocal elastica and we study the difference with Challamel's proposal.

More precisely, Eringen's nonlocal elastica equation does not possess a Lagrangian formulation. As a consequence, we do not have any structure at hand that can constrain the construction of a discrete analogue by a discrete embedding.

In order to solve this problem, we use in Chapter VII the notion of **variational integrating factor** *that is a non zero function ψ, such that the equation multiplied by ψ becomes variational, to construct a Lagrangian formulation of Eringen's nonlocal elastica. This enables us to provide also a Hamiltonian structure.*

We then use the discrete variational embedding of Lagrangian or Hamiltonian systems studied in Part A to define a discrete analogue of Eringen's nonlocal elastica.
It must be noted that this discrete model does not coincide with Challamel's model.

How to choose between Challamel's discrete model and our model ?

In Chapter VIII, We compare the two with respect to the preservation of the value of the first integral obtained via the Hamiltonian structure. We prove that our model is more efficient from this point of view.

Mechanical problems lead to more complicated situations than the ones discussed for Eringen's nonlocal elastica. In particular, for microstructured media, we have different scales that play different roles in the modeling and a mixing between discrete and continuous modeling is unavoidable.

In Chapter IX, we discuss the problem of modeling deformation of a graphene membrane for which such a problem exists. We propose to use time-scale for the modeling of such a situation.

The following articles have been extracted of this manuscript:

1. B. Anerot, J. Cresson, K. Hariz-Belgacemn, F. Pierret, Noether's type theorem on time scales, J. Math. Phys. 61, 113502 (2020), 32.p
2. J. Cresson, K. Hariz-Belgacem, About the structure of the discrete and continuous Eringen's nonlocal Elastica, Mathematics and Mechanics of Solids, 2022, 24.p, In press
3. J. Cresson, K. Hariz-Belgacem, A. Szafranska, High order time scale calculus and Galerkin's variational integrators, 2022, preprint, 11.p
4. J. Cresson, K. Hariz-Belgacem, A. Hamdouni, J. Palafox, Noether's theorem for Hamiltonian systems on time scales, preprint, 2022, 24.p
5. J. Cresson, K. Hariz-Belgacem, F. Pierret, Discrete embedding formalisms and applications, preprint, 20.p, 2022.

Notation

General Notations

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$ (resp. $\mathbb{R}^-$)	Set of non-negative (resp. non-positive) real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}^*$	Set of non-zero natural numbers
$\mathbb{Z}$	Set of integer numbers
$a,\ b$	Real numbers with $a < b$
$x \cdot y$	Scalar product of two vectors $x, y \in \mathbb{R}^d$
$\\| \cdot \\|$	Euclidean norm in $\mathbb{R}^d$
$(\cdot)^\mathsf{T}$	Matrix transpose
Id	Identity function
$\mathscr{C}([a,b],\mathbb{R}^d)$	Set of continuous functions over $[a,b]$ with values in $\mathbb{R}^d$
$\mathscr{C}^k([a,b],\mathbb{R}^d)$	Set of k-times continuously differentiable functions ver $[a,b]$ with values in $\mathbb{R}^d$
L	Lagrangian, i.e., a continuous and $\mathscr{C}^2$-function with respect to the last two variables
$\mathcal{L}$	Lagrangian functional associated to L
$D\mathcal{L}(x)(w)$	Fréchet differential of $\mathcal{L}$ at x along the direction w
x	Trajectories of dynamical systems, i.e., $x : t \mapsto \mathbb{R}^d$
$\dot{x}$	Time derivative (or dot derivative) of x , i.e., $\dot{x} = dx/dt$
$\ddot{x}$	Second time derivative of x , i.e., $\ddot{x} = d^2x/dt^2$

Notations related to time scales calculus

$\mathbb{T}$	Time scales, i.e., an arbitrary non-empty closed subset of $\mathbb{R}$
$\mathbb{T}_\kappa$	$\mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$
$\mathbb{T}^\kappa$	$\mathbb{T} \setminus (\sup \mathbb{T}, \rho(\sup \mathbb{T})]$
$\mathbb{T}^\kappa_\kappa$	$\mathbb{T}_\kappa \cap \mathbb{T}^\kappa$
$\mathbb{T}^{\kappa^2}$	$(\mathbb{T}^\kappa)^\kappa$
σ	Forward jump operator defined by $\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}$

ρ	Backward jump operator defined by $\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}$
x^σ (resp. x^ρ)	$x \circ \sigma$ (resp. $x \circ \rho$)
μ	Forward graininess function defined by $\mu(t) = \sigma(t) - t$
ν	Backward graininess function defined by $\nu(t) = t - \rho$
Δx (resp. ∇x)	Δ-derivative (resp. ∇-derivative) of a function x on $\mathbb{T}$
LD (resp. LS)	Set of all left-dense (resp. left-scattered) points of $\mathbb{T}$.
RD (resp. RS)	Set of all right-dense and (resp. right-scattered) points of $\mathbb{T}$.
$\mathrm{C}^0(\mathbb{T})$ (or simply $\mathrm{C}(\mathbb{T})$)	Functional spaces of continuous functions on $\mathbb{T}$
$\mathrm{C}^0_{\mathrm{rd}}(\mathbb{T})$	Functional spaces of rd-continuous functions on $\mathbb{T}$
$\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$	Functional spaces of Δ-differentiable functions on $\mathbb{T}_\kappa$ with rd-continuous ∇-derivative
$\mathrm{C}^0_{\mathrm{ld}}(\mathbb{T})$	Functional spaces of ld-continuous functions on $\mathbb{T}$
$\mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$	Functional spaces of ∇-differentiable functions on $\mathbb{T}_\kappa$ with ld-continuous Δ-derivative
$\int \Delta\tau$	Cauchy Δ-integral
$\mathbb{T}^*$	Dual time scale of $\mathbb{T}$ defined by $\mathbb{T}^* := \{\tau \in \mathbb{R} :\ -\tau \in \mathbb{T}\}$
f^*	Dual function defined on $\mathbb{T}^*$ by $f^*(\tau) = f(-\tau)$ for all $\tau \in \mathbb{T}^*$
$\widehat{\Delta}$ (resp. $\widehat{\nabla}$)	Δ-derivative (resp. ∇-derivative) associated to $\mathbb{T}^*$
$\hat\sigma,\ \hat\rho,\ \hat\mu,\ \hat\nu$	Forward jump operator, backward operator, forward graininess and backward graininess associated to $\mathbb{T}^*$

Specific notations in Chapter II

$n,\ N$	Non-zero natural numbers
$\mathbb{T}$	Usual time scales, i.e., $\mathbb{T} = \{t_i = a + ih, i = 0, \dots, N\}$ with $h = (b-a)/N$
$\mathbb{T}_+$ (resp. $\mathbb{T}_-$)	$\mathbb{T}_+ = \mathbb{T} \setminus \{t_N\}$ (resp. $\mathbb{T}_- = \mathbb{T} \setminus \{t_0\}$)
$\mathbb{T}^\pm$	$\mathbb{T}^+ \cap \mathbb{T}^-$
I	Subinterval of $[a,b]$
$I_\mathbb{T}$	$I_\mathbb{T} = \mathbb{T} \cap I$
$\mathcal{F}([a,b],\mathbb{R}^d)$	Set of functions over $[a,b]$ with values in $\mathbb{R}^d$
$\mathcal{F}(\mathbb{T},\mathbb{R})$	Set of functions over $[a,b]$ with values in $\mathbb{T}$
$\mathcal{F}_0(\mathbb{T},\mathbb{R})$	Subset of $\mathcal{F}(\mathbb{T},\mathbb{R})$, i.e., $\mathcal{F}_0(\mathbb{T},\mathbb{R}) = \{G \in \mathcal{F}(\mathbb{T},\mathbb{R}),\ G_0 = G_N = 0\}$
X	Discrete function, i.e., $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ where $X_i = x(t_i),\ i = 0, \dots, N$
$\mathbb{1}_A$	Indicator function of a subset A, i.e., $\mathbb{1}_A(t) = 1$ if $t \in A$ and 0 otherwise.
$\mathscr{P}_n([a,b],\mathbb{R})$	Set of continuous functions that are piecewise polynomial functions of degree n
κ	Interpolation map
$\langle\cdot,\cdot\rangle_\star$	Discrete scalar product on $\mathcal{F}(\mathbb{T},\mathbb{R}^d)$
π	Restriction map $\pi : \mathcal{F}(\mathbb{T},\mathbb{R}) \to \mathcal{F}(I_\mathbb{T},\mathbb{R})$ defined by $\pi(x) = X$ where $X_i = x(t_i),\ t_i \in I_\mathbb{T}$
Δ (resp. ∇)	Discrete Δ-derivative (resp. ∇-derivative)
J_Δ (resp. J_∇)	Discrete Δ-antiderivative (resp. ∇-antiderivative)

Specific notations in Chapter III

$\mathbb{T}_\eta$	Control time scale over $[0,1]$
$\mathbb{T}_{i,C}$	$\mathbb{T}_C = \bigcup_{i=0}^{n-1} \mathbb{T}_{i,C}$, $i = 1,\dots,N-1$
$\mathbb{T}_C$	Time scale over $I_i^\pm$ corresponds to $\mathbb{T}_\eta$
$\mathbb{T}_q$	Quadrature time scale over $[0,1]$
$\mathbb{T}_{i,Q}$	Time scale over I_i corresponds to $\mathbb{T}_q$
$\mathbb{T}_Q$	$\mathbb{T}_Q = \bigcup_{i=0}^{n-1} \mathbb{T}_{i,Q}$, $i = 1,\dots,N-1$
X	Continuous function on $\mathbb{T}$, i.e., $X \in \mathrm{C}(\mathbb{T},\mathbb{R})$
X_c	Control function defined on $\mathbb{T}_C$
Z	Extension of X defined on $\mathbb{T}\cup\mathbb{T}_C$ by $Z_{\vert_{\mathbb{T}}} = X$ and $Z_{\vert_{\mathbb{T}_C}} = C$
κ_m	Interpolation map, $k_m : \mathscr{C}(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}) \longrightarrow \mathscr{P}_m([a,b],\mathbb{R})$
w	Weight function defined over $\mathbb{T}_q$
$\mathscr{C}_0(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$	Set of variations of $\mathcal{L}_w$, i.e., subset of $\mathscr{C}(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$ for functions vanish at a and b
$\int \Delta_{q,w} s$	$(\mathbb{T}_q, w)$-integral of a function f over $\mathbb{T}$

Specific notations in Chapter IV

$\{g_s\}_{s\in\mathbb{R}}$	Family of a one-parameter group of diffeomorphisms acts on $[a,b]\times\mathbb{R}^d$
$\zeta,\ \xi$	Infinitesimals associated to $\{g_s\}_{s\in\mathbb{R}}$
$\mathbf{X}$	Infinitesimal generator associated to $\{g_s\}_{s\in\mathbb{R}}$
H	Hamiltonian function
$\mathcal{L}_H$	Hamiltonian functional associated to H
$\{\phi_s\}_{s\in\mathbb{R}}$	Canonical group of projectable transformations acts on $[a,b]\times\mathbb{R}^{2d}$
G	Generating function associated to $\{\phi_s\}_{s\in\mathbb{R}}$
$\{\cdot,\cdot\}$	Poisson bracket

Specific notations in Chapter V and Chapter VI

$\mathcal{L}_{\Delta,\mathbb{T}}$ (resp. $\mathcal{L}_{\nabla,\mathbb{T}}$)	Lagrangian functional in the framework of the nonshifted Δ- (resp. ∇-) calculus of variations
$\mathcal{L}^\sigma_{\Delta,\mathbb{T}}$ (resp. $\mathcal{L}^\rho_{\nabla,\mathbb{T}}$)	Lagrangian functional in the framework of the shifted Δ- (resp. ∇-) calculus of variations
$\mathcal{L}_{H,[a,b]_\mathbb{T}}$	Hamiltonian functional in the framework of the nonshifted Δ-calculus of variations
$\mathcal{L}^\sigma_{H,[a,b]_\mathbb{T}}$ (resp. $\mathcal{L}^\rho_{H,[a,b]_\mathbb{T}}$)	Hamiltonian functional in the framework of the shifted Δ- (resp. ∇-) calculus of variations
$\mathscr{H}$	Hamiltonian function defined on $\mathbb{T}$ associated to the Lagrangian L defined by $\mathscr{H}(t,x,v) = -L(t,x,v) + v\cdot\partial_v L(t,x,v) + \mu(t)\partial_t(t,x,v)$

Part A

Discrete Embedding Formalisms and High-order Time-scale Calculus

Chapter I

Brief Overview on The Time Scales Calculus

This chapter includes definitions and properties related to time scales calculus (see [1], [11], [12], [13]) and Lagrangian systems (see [4], [17], [60], [89]). In particular, we remind the algebraic properties of the derivatives and antiderivatives in time scales calculus, like the chain rule formula, the Leibniz property or the change of variable which will be of constant uses in the next Chapters.

I.1 Introduction

The time scales calculus is a modern and an efficient theory that is able to deal with discrete, continuous or mixed processes using a unique formalism. The time scales calculus dates back to the original work of Stephan Hilger under his supervisor Bernd Aulbach [52] in 1988. It has found widespread applications in many science areas such as statistics, biology, economics [6], finance, engineering, physics, etc.

A time scales $\mathbb{T}$ is an arbitrary non-empty closed subset of $\mathbb{R}$. The well known examples of time scales are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ which cover the classical continuous and discrete setting respectively. Further applications, the choice of $\mathbb{T} = q^{\mathbb{N}}$ is exploited in quantum setting [59] while $\mathbb{T} = [0, 1] \cup h\mathbb{Z}$ is used in a mixed setting such as the Euler-Bernoulli continuous beam problem [22].

To study each setting using separate formulae is difficult, therefore, *unification* and *extension* are the two key features of the time scales calculus.

In [52], S. Hilger introduced a new definition of derivatives for functions defined over $\mathbb{T}$ denoted by Δ- and ∇-derivatives. He unifies sums and integrals by introducing the notion of Δ- and ∇-integrals.

From the algebraic viewpoint, the usual formulas (e.g. Leibniz, quotient and integral by parts rules, etc.) remain inherited at the time scales setting due to the notions of Δ- and ∇-derivatives as well as Δ- and ∇-integrals.

Organization of the chapter. In Section I.2, we remind some definitions and notations about time scales and give some particular statements about the chain rule formula and the substitution formula for Δ- and ∇-derivatives in the time scales setting, as well as the corresponding Leibniz formula. In Section I.3, we give recalls on Lagrangian systems and the classical calculus of variations.

I.2 Preliminaries on time scales

In what follows, we will denote by $\mathbb{T}$ a time scales, i.e., an arbitrary non-empty closed subset of $\mathbb{R}$.

Two operators play a central role studying time scales, *the backward* and *forward jump operators.*

Definition I.1. *The backward and forward jump operators* $\rho, \sigma : \mathbb{T} \longrightarrow \mathbb{T}$ *are respectively defined by:*

$$\forall t \in \mathbb{T},\ \rho(t) = \sup\{s \in \mathbb{T},\ s < t\}, \quad \sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}, \tag{I.2.1}$$

where we put $\sup \emptyset = \inf \mathbb{T}$ *and* $\inf \emptyset = \sup \mathbb{T}$.

Definition I.2. *A point* $t \in \mathbb{T}$ *is said to be left-dense (resp. left-scattered, right-dense and right-scattered) if* $\rho(t) = t$ *(resp.* $\rho(t) < t$, $\sigma(t) = t$ *and* $\sigma(t) > t$*).*

Let LD (resp. LS, RD and RS) denote the set of all left-dense (resp. left-scattered, right-dense and right-scattered) points of $\mathbb{T}$.

Definition I.3. *The graininess and backward graininess functions* $\mu, \nu : \mathbb{T} \longrightarrow \mathbb{R}^+$ *are respectively defined by:*

$$\forall t \in \mathbb{T},\ \mu(t) = \sigma(t) - t, \quad \nu(t) = t - \rho(t) \tag{I.2.2}$$

Example I.1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) = 0$. If $\mathbb{T} = hZ,\ h > 0$, then $\sigma(t) = t + h,\ \rho(t) = t - h$, and $\mu(t) = \nu(t) = h$. On the other hand, if $\mathbb{T} = \{2^n,\ \mathbb{N} \cup \{0\}\}$, and

- when $n = 0$, then $t = 1$ and $\sigma(t) = 2t, \quad \rho(t) = t, \quad \mu(t) = t, \quad \nu(t) = 0,$
- when $n \neq 0$, then $t = 2^n$ and $\sigma(t) = 2t, \quad \rho(t) = \frac{1}{2}t, \quad \mu(t) = t, \quad \nu(t) = \frac{1}{2}t.$

We denote by $\mathbb{T}_\kappa = \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$, $\mathbb{T}^\kappa = \mathbb{T} \setminus (\sup \mathbb{T}, \rho(\sup \mathbb{T})]$ and $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

I.2.1 The Δ and ∇ derivatives

Let us recall the usual definitions of Δ and ∇-differentiability.

Definition I.4. *A function* $u : \mathbb{T} \longrightarrow \mathbb{R}^n$ *, where* $n \in \mathbb{N}$*, is said to be* Δ*-differentiable at* $t \in \mathbb{T}^\kappa$ *(resp.* ∇*-differentiable at* $t \in \mathbb{T}_\kappa$*) if the following limit exists in* $\mathbb{R}^n$*:*

$$\lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} \left(\textit{resp.} \lim_{\substack{s \to t \\ s \neq \rho(t)}} \frac{u(s) - u(\rho(t))}{s - \rho(t)} \right). \tag{I.2.3}$$

In such a case, this limit is denoted by $\Delta u(t)$ *(resp.* $\nabla u(t)$*).*

Example I.2. If $\mathbb{T} = \mathbb{R}$, then $\Delta x(t) = \nabla x(t) = \dot{x}(t)$, the Δ- and ∇-derivatives coincide with the usual one. If $\mathbb{T} = h\mathbb{Z}$, then

$$\Delta x(t) = \frac{x(t+h) - x(t)}{h} := \Delta_+ x(t) \quad \text{and} \quad \nabla x(t) = \frac{x(t) - x(t-h)}{h} := \Delta_- x(t),$$

that is the Δ- and ∇-derivatives coincide with the usual forward and backward discrete derivatives respectively.

The characterization of constant of motion is related to the following fundamental result in [11, Corollary 1.68, p.25]

Proposition I.1. *Let* $u : \mathbb{T} \longrightarrow \mathbb{R}^n$ *. Then,* u *is* Δ*-differentiable on* $\mathbb{T}^\kappa$ *with* $\Delta u = 0$ *if and only if there exists* $c \in \mathbb{R}^n$ *such that* $u(t) = c$ *for every* $t \in \mathbb{T}$.

The analogous results for ∇-differentiability are also valid.

I.2.2 Some functional spaces

Definition I.5. *A function u is said to be rd-continuous (resp. ld-continuous) on $\mathbb{T}$ if it is continuous at every $t \in \mathrm{RD}$ (resp. $t \in \mathrm{LD}$) and if it admits a left-sided (resp. right-sided) limit at every $t \in \mathrm{LD}$ (resp. $t \in \mathrm{RD}$).*

We respectively denote by $\mathrm{C}^0_{\mathrm{rd}}(\mathbb{T})$ and $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ the functional spaces of rd-continuous functions on $\mathbb{T}$ and of Δ-differentiable functions on $\mathbb{T}^\kappa$ with rd-continuous Δ-derivative. Similarly, we denote by $\mathrm{C}^0_{\mathrm{ld}}(\mathbb{T})$ and $\mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$, respectively, the functional spaces of ld-continuous functions on $\mathbb{T}$ and of ∇-differentiable functions on $\mathbb{T}_\kappa$ with ld-continuous ∇-derivative.

I.2.3 The Δ- and ∇-integrals

Let us denote by $\int \Delta\tau$ the Cauchy Δ-integral defined in [11, p.26] with the following result [11, Theorem 1.74, p.27]:

Theorem I.1. *For every $u \in \mathrm{C}^0_{\mathrm{rd}}(\mathbb{T}^\kappa)$, there exist a unique Δ-antiderivative U of u in sense of $\Delta U = u$ on $\mathbb{T}^\kappa$ vanishing at $t = a$. In this case the Δ-integral is defined by*

$$U(t) = \int_a^t u(\tau)\Delta\tau \quad \textit{for every} \quad t \in \mathbb{T}.$$

Similarly, we denoted by $\int \nabla\tau$ the Cauchy ∇-integral defined in [11, Theorem 8.45, p.332]:

Theorem I.2. *For every $u \in \mathrm{C}^0_{\mathrm{ld}}(\mathbb{T}_\kappa)$, there exist a unique ∇-antiderivative W of u in sense of $\Delta W = u$ on $\mathbb{T}_\kappa$ vanishing at $t = a$. In this case the ∇-integral is defined by*

$$W(t) = \int_a^t u(\tau)\nabla\tau \quad \textit{for every} \quad t \in \mathbb{T}.$$

Example I.3. Let $a \leq b$, and let $[a, b] \subset \mathbb{T}$.

- If $\mathbb{T} = \mathbb{R}$, then the Δ- and ∇-integrals coincide with the usual Riemann integral, i.e.

$$\int_a^b x(t)\ \Delta t = \int_a^b x(t)\ \nabla t = \int_a^b x(t)\ dt$$

- If $\mathbb{T} = h\mathbb{Z}$, then the Δ- and ∇-integrals coincide with the usual Riemann integral, i.e.

$$\int_a^b x(t)\ \Delta t = h \sum_{t \in [a,b] \cap \mathbb{T}^\kappa} f(t) \quad \text{and} \quad \int_a^b x(t)\ \nabla t = h \sum_{t \in [a,b] \cap \mathbb{T}_\kappa} f(t)$$

I.2.4 Algebraic properties of Δ- and ∇- derivatives

The Δ-derivative satisfies a Leibniz formula (see [11, Corollary 1.20, p.8]) in the following theorem:

Theorem I.3 (Leibniz formula for the Δ-derivative)**.** *Let $v, w : \mathbb{T} \longrightarrow \mathbb{R}^n$. If v and w are Δ-differentiable at $t \in \mathbb{T}^\kappa$, then the scalar product $v \cdot w$ is Δ-differentiable at t and the following Leibniz formula holds:*

$$\begin{aligned} \Delta\left(v \cdot w\right)(t) &= v^\sigma(t) \cdot \Delta w(t) + \Delta v(t) \cdot w(t), \\ &= v(t) \cdot \Delta w(t) + \Delta v(t) \cdot w^\sigma(t). \end{aligned} \tag{I.2.4}$$

We have a time scales Leibniz formula for the ∇-derivative (see [14, Proposition 7]).

Theorem I.4 (Leibniz formula for ∇-derivative). *Let* $v, w : \mathbb{T} \longrightarrow \mathbb{R}^n$ *and* $t \in \mathbb{T}^{\kappa}_{\kappa}$. *If the following properties are satisfied:*

- *σ is ∇-differentiable at t,*
- *v is Δ-differentiable at t,*
- *w is ∇-differentiable at t,*

then, $v^{\sigma} \cdot w$ is ∇-differentiable at t and the following Leibniz formula holds:

$$\nabla \left(v^{\sigma} \cdot w\right)(t) = v(t) \cdot \nabla w(t) + \nabla\sigma(t) \cdot \Delta v(t) \cdot w(t). \tag{I.2.5}$$

We refer to [14, Section 3.2, p.550] for many examples of ∇-differentiable σ as well as a discussion of the restrictions on a time scale imposed by such a condition.

I.2.5 Chain rule formula and the substitution formula

We have a time scales chain rule formula (see [11, Theorem 1.93]).

Theorem I.5 (Time scales chain rule). *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. Let* $w : \widetilde{\mathbb{T}} \longrightarrow \mathbb{R}$. *If* $\Delta v(t)$ *and* $\Delta_{\widetilde{\mathbb{T}}} w(v(t))$ *exist for* $t \in \mathbb{T}^{\kappa}$, *then*

$$\Delta \left(w \circ v\right) = \left(\Delta_{\widetilde{\mathbb{T}}} w \circ v\right) \Delta v \tag{I.2.6}$$

With the time scales chain rule, we obtain a formula for the derivative of the inverse function (see [11, Theorem 1.97]).

Theorem I.6 (Derivative of the inverse). *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. Then*

$$\frac{1}{\Delta v} = \Delta_{\widetilde{\mathbb{T}}} \left(v^{-1}\right) \circ v \tag{I.2.7}$$

at points where Δv is different from zero.

Another formula from the chain rule is the substitution rule for integrals (see [11, Theorem 1.98]).

Theorem I.7 (Substitution). *Assume that* $v : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := v(\mathbb{T})$ *is a time scales. If* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is a rd-continuous function and v is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\Delta v(t)\Delta t = \int_{v(a)}^{v(b)} \left(f \circ v^{-1}\right)(s)\Delta_{\widetilde{\mathbb{T}}} s. \tag{I.2.8}$$

I.3 Reminder about the Lagrangian calculus of variations

In this section, we briefly present some results concerning Lagrangian systems.

The calculus of variation is one of the classical branches of mathematics. It is a technique that used to find critical points (or extremals) of functional defined on infinite dimensional spaces.

It was many concrete geometry and physics problems developed in the 17^{th} century which unified due to the calculus of variation. This method has been named by Euler starting from the so-called *brachistochrone problem*. The analysis of this problem was posed by John Bernoulli in 1696 and was solved by John Bernoulli, James Bernoulli, Newton, and L'Hospital. This method become a mathematical approach due to *Euler and Lagrange* in 1755, they discovered a pragmatic approach and provided a necessary condition in terms of the first variation. The result is the *Euler-Lagrange equation of motion*. We refer the reader to ([17], [18], [33], [60], [89]) and references therein for the history of the calculus of variations.

I.3.1 Lagrangian systems

A *Lagrangian function* L (or simply a *Lagrangian*) is a function defined by

$$\begin{array}{rccc} L: & [a,b]\times\mathbb{R}^n\times\mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (t,x,v) & \longmapsto & L(t,x,v), \end{array} \tag{I.3.1}$$

such that L is continuous and of class $\mathscr{C}^2$ with respect to the last two variables.

The Lagrangian L defines the so-called *Lagrangian functional* $\mathcal{L}$ is given by

$$\begin{array}{rccc} \mathcal{L}: & \mathscr{C}^1\left([a,b],\mathbb{R}^d\right) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\int_a^b L(t,x(t),\dot{x}(t))\,dt \end{array}$$

where x is a function called trajectory or curve, the dot indicates the derivative with respect to t.

The problem is to find the trajectory for which the functional $\mathcal{L}$ is critical (or stationary), that is to find the critical point of the variational problem [17, p.29]

$$\mathcal{L}(x)=\int_a^b L(t,x(t),\dot{x}(t))\,dt \quad \text{such that} \quad x(a)=\alpha, \quad x(b)=\beta.$$

This strategy can be done by using the calculus of variations.

Definition I.6 (First variation)**.** *The Fréchet differential of $\mathcal{L}$ at x along the direction $w\in\mathscr{C}^1\left([a,b],\mathbb{R}^d\right)$ is defined by*

$$D\mathcal{L}(x)(w):=\lim_{\epsilon\to 0}\frac{\mathcal{L}(x+\epsilon w)-\mathcal{L}(x)}{\epsilon}.$$

The following definition gives a necessary condition for the functional $\mathcal{L}$ to have critical points.

Definition I.7. *A function $x\in\mathscr{C}^1\left([a,b],\mathbb{R}^d\right)$ is a critical point for the functional $\mathcal{L}$ if*

$$D\mathcal{L}(x)(w)=0, \quad \textit{for all} \quad w\in\mathscr{C}^1\left([a,b],\mathbb{R}^d\right).$$

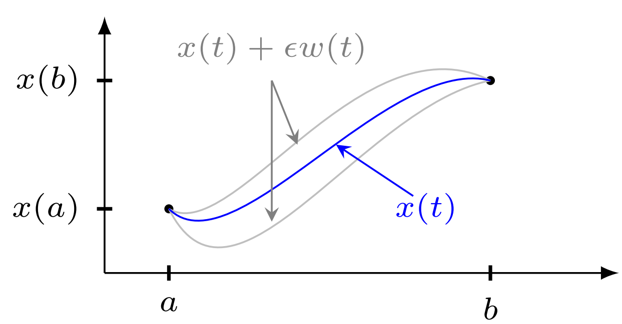


Figure I.1: The simplest variational problem.

I.3.2 Euler–Lagrange equations

Critical points of the functional $\mathcal{L}$ can be characterised by a second-order differential equation called the Euler-Lagrange equation. This based on the fundamental lemma of the calculus of variations called the DuBois–Reymond lemma (see [60, Lemma 1.3.1, p.9]).
Now we introduce the space of variations

$$\mathscr{C}_0^1\left([a,b],\mathbb{R}^d\right) := \left\{ w \in \mathscr{C}^1([a,b],\mathbb{R}^d),\ w(a) = w(b) = 0 \right\}.$$

Lemma I.1 (The fundamental lemma of calculus of variations). *Let $f \in \mathscr{C}\left([a,b],\mathbb{R}^d\right)$ and*

$$\int_a^b f(t)w(t) = 0 \quad \textit{for all} \quad w \in \mathscr{C}_0^1\left([a,b],\mathbb{R}^d\right),$$

then $f(t) = 0$ for all $t \in [a,b]$.

Theorem I.8 (Variational principle). *Let $x \in \mathscr{C}^1([a,b],\mathbb{R}^d)$ be a critical point of the functional $\mathcal{L}$. Then, x is solution of the Euler-Lagrange equation given by:*

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\cdot, x, \dot{x})\right](t) = \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)), \quad \forall t \in [a,b]. \tag{I.3.2}$$

It is worth mentioning that every solution of the Euler-Lagrange equation is also a solution of the so-called second Euler-Lagrange equation [89, Proposition 6.3, p.154].

Theorem I.9 (The second Euler-Lagrange equation). *Let $x \in \mathscr{C}^1([a,b],\mathbb{R}^d)$ be a critical point of the functional $\mathcal{L}$. Then, x is solution of the Euler-Lagrange equation given by:*

$$\frac{d}{dt}\left[\dot{x} \cdot \frac{\partial L}{\partial v}(\cdot, x, \dot{x}) - L(\cdot, x, \dot{x})\right](t) = -\frac{\partial L}{\partial t}(t, x(t), \dot{x}(t)), \quad \forall t \in [a,b]. \tag{I.3.3}$$

We note that if the Lagrangian L does not depend explicitly on time, the second Euler-Lagrange equation reduces to

$$\frac{d}{dt}\left[\dot{x} \cdot \frac{\partial L}{\partial v}(\cdot, x, \dot{x}) - L(\cdot, x, \dot{x})\right](t) = 0.$$

Meaning that the quantity $\dot{x} \cdot \frac{\partial L}{\partial v}(t, x, \dot{x}) - L(t, x, \dot{x})$ is constant over all solutions of (I.3.2).

Example I.4 (The brachistochrone problem - Johann Bernoulli, 1696). The problem is to find the curve joining two points $P_a = (0,0)$ and $P_b = (b, B)$, along which a particle falling from rest under the influence of gravity travels from the higher to the lower point in the least time (see [17, p.7]).

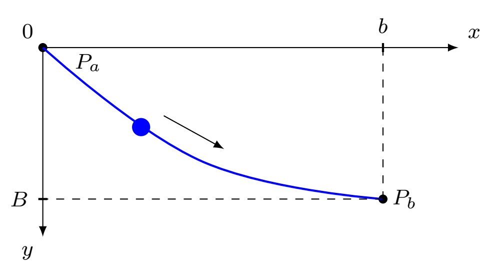


Figure I.2: The brachistochrone problem.

This problem is equivalent to a function $y \in \mathscr{C}^1([0,b],\mathbb{R})$ that is extremal of the problem

$$\mathcal{L}(y) = \int_0^b L(x,y,\dot{y})\,dx = \lambda \int_0^b \frac{\sqrt{1+\dot{y}(x)}}{y(x)}\,dx, \quad \text{such that} \quad y(0)=0, \quad y(b)=P,$$

where $\lambda > 0$. Since L does not depend explicitly on x, that is, $L = L(y,v)$, then the corresponding Euler-Lagrange equation I.3.2 reduces to

$$\frac{\partial L}{\partial v}(y,\dot{y}) = \text{constant} \quad \Longrightarrow \quad L(y,\dot{y})\frac{\dot{y}}{1+\dot{y}^2} = \text{constant}.$$

The second Euler-Lagrange equation I.3.3 reduces to

$$H(y,\dot{y}) := L(y,\dot{y}) - \dot{y}\cdot\frac{\partial L}{\partial v}(y,\dot{y}) = \text{constant} \quad \Longrightarrow \quad y\left(1+\dot{y}^2\right) = \mu, \quad \mu > 0.$$

Hence, the function $H(y,\dot{y})$ is first integral for the brachistochrone problem and the solution curve is a class of curves called *cycloids*.

Chapter II

Classical Discrete Embeddings of Ordinary Differential Equations

In this chapter, we define an abstract framework called discrete finite differences embedding which can be used to obtain discrete analogue of formal functional relations in the spirit of category theory. For ordinary differential equations we exhibit three main discrete associate : the differential, integral or variational discrete embeddings which corresponds to classical numerical scheme including variational integrators.

This Chapter is based on a preprint "Discrete embeddings of ordinary differential equations" *with J. Cresson and F. Pierret from Observatoire de Paris which can be seen as a more general and abstract version of the embedding formalisms proposed in [29], [30] and initiated in [16], [24]–[26].*

II.1 Introduction

Embedding formalism was initiated in [16], [24], [25]. It is a strategy used to obtain analogue of the classical derivative, anti-derivative, functionals, etc, in a more general framework. As a consequence, such a formalism can be used to obtain extension of classical ordinary differential equations or partial differential equations. Embedding formalisms are algebraic in nature and give a precise connection with the classical differential calculus.

The *discrete setting* of this formalism was developed in [26], [29], [30]. An account of this formalism for problems related to discretization was depicted in [26], [28] and L. Bourdin ([13], [14]) using to the time-scale calculus.

Organization of the chapter. In Section II.2, we define a more general and abstract version of the embedding formalisms developed in [24]. Section II.3, Section II.5 and II.6 contain the definition of the discrete analogue of continuous objects like functions and differential or integral operators. The main point is to give an explicit connection between the discrete and continuous case. This is done using some particular mappings that we call discretization and interpolation in the following. As a consequence, in Section II.7 we are able to give without any computations on sums or classical methods of rearranging terms new formulations of classical results (discrete integration by parts, discrete fundamental theorem of differential calculus, etc.). In Sections II.8 and II.9, we use the previous formalism to define the discrete embedding of a formal functional. This abstract setting allows us to cover very different objects like differential equations, integral equations or Lagrangian functional. Then, we describe the three natural discrete way to obtain a discrete analogue of a differential equation: the differential, integral and variational case. Each of these procedure leads to different discrete realization of the same equations.

II.2 Embedding formalisms

Let E be an infinite dimensional functional space and let F be a finite dimensional vector space.

Definition II.1. *Let $\pi : E \to F$ be a surjective projection. There is an application $\kappa : F \to E$, called a lift map, such that*

$$\kappa \circ \pi = \mathrm{Id}_F \quad \textit{and} \quad \pi \circ \kappa = \mathrm{Id}_E$$

Definition II.2. *Let E_1 and E_2 be infinite two dimensional functional spaces and let F_1 and F_2 be two finite dimensional vector spaces. Let $\pi_1 : E_1 \to F_1$ and $\pi_2 : E_2 \to F_2$ be two lift maps and let $A : \mathrm{Dom}(A) \subseteq E_1 \to E_2$ be a linear operator defined on $\mathrm{Dom}(A)$. The lift map π_1 is said to be compatible if and only if*

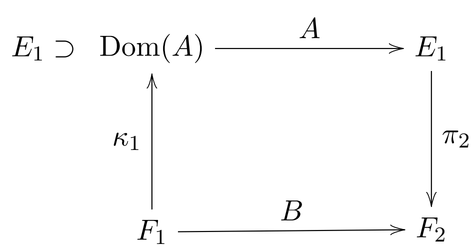
$$\kappa_1(F_1) \subset \mathrm{Dom}(A), \tag{II.2.1}$$

where π_1 is the projection from E_1 onto F_1.

The finite projection of A is the operator B defined on F_1 by

$$B = \pi_2 \circ A \circ \kappa_1.$$

Thus, we have the following commutative diagram



Using this definition and taking for A the classical derivative d/dt (or an operator associated to the classical derivative) or the antiderivative $\int_{\cdot}^{t} \cdot \, dt$ and for F an appropriate vector space, one can defined on these new set an extended derivative D and a new antiderivative $I_{\cdot}^{t}$ acting on F. Using these operators, a differential equation $F(x, dx/dt, \ldots, d^n x/dt^n) = 0$ is embedded over F via the formula $F(X, D[X], \ldots, D^n[X]) = 0$ as long as this equations keep sense.

In the following, we illustrate this construction in the case of a discrete functional space.

II.3 Direct discrete embeddings

Let $N \in \mathbb{N}^*$ and let $\mathbb{T} = \{t_i = a + ih,\ i = 0, \ldots, N\}$ be the usual time scales. Let $n \in \mathbb{N}$ and we assume that N is a multiple of n, i.e., there exist $p \in \mathbb{N}^*$ such that $N = np$. We will use here the notation $I_{i,n} = [t_{(n+1)i}, t_{(n+1)(i+1)}]$, $I^+_{i,n} = [t_{(n+1)i}, t_{(n+1)(i+1)}[$ and $I^-_{i,n} =]t_{(n+1)i}, t_{(n+1)(i+1)}]$ as subintervals of $[a, b]$. Thus, we have

$$\bigcup_{i=1}^{p} I_{i,n} = [a, b].$$

We denote by $\mathscr{P}_n([a, b], \mathbb{R})$ the set of continuous functions which are piecewise polynomial functions of degree n over subintervals $I_{i,n} = [t_{ni},\ t_{n(i+1)}]$. We denote also by $\mathscr{P}^+_n([a, b], \mathbb{R})$ (resp. $\mathscr{P}^-_n([a, b], \mathbb{R})$) the set of piecewise polynomial functions of degree n over $I^+_{i,n}$ (resp. $I^-_{i,n}$) which are right continuous (resp. left continuous), for $i = 1, \dots, p$.

Lagrange basis polynomials. Let $T = T_0, \dots, T_n \in [a, b]$. A basis of $\mathscr{P}_n([a, b], \mathbb{R})$ is given by the Lagrange polynomials defined as follows (see [34, p.24] or [35, p.21-22])

$$\ell_i(t) = \prod_{\substack{i=0\\ j\neq i}}^{n} \frac{(t - T_j)}{(T_i - T_j)}, \quad i = 0, \dots, n$$

for all $t \in [a, b]$.

Interpolation map of degree n. For all $P \in \mathscr{P}_n([a, b], \mathbb{R})$, there exists a unique $X = (X_0, \dots, X_n) \in \mathbb{R}^{n+1}$ such that

$$\forall t \in [a, b], \quad P(t) := P^X_{n,\ell}(t) = \sum_{i=0}^{n} X_i\ \ell_i(t).$$

It is the well-known Lagrange interpolation polynomial of degree n associated to X. Thus, we have an identification between $\mathscr{P}_n$ and $\mathbb{R}^{n+1}$, i.e., $P^X_{n,\ell} \sim \{X_i\}^n_{i=0}$.

II.4 Continuous versus discrete functions

II.4.1 Discrete functions and discretization

Let us begin with a general definition:

Definition II.3 (Discrete function)**.** *A discrete function is an element $X \in \mathcal{F}(\mathbb{T}, \mathbb{R})$.*

By definition, a discrete function is completely characterized by the finite set $X_i = X(t_i)$, $i = 0, \dots, N$.

In the following, we illustrate all the discrete notions with a single example given by the following discrete function: $X = \{2, 1, 3, 2, 7, 5, 2\} \in \mathcal{F}(\mathbb{T}, \mathbb{R})$ with $\mathbb{T} = \{0, 1, 2, 3, 4, 5, 6\}$.

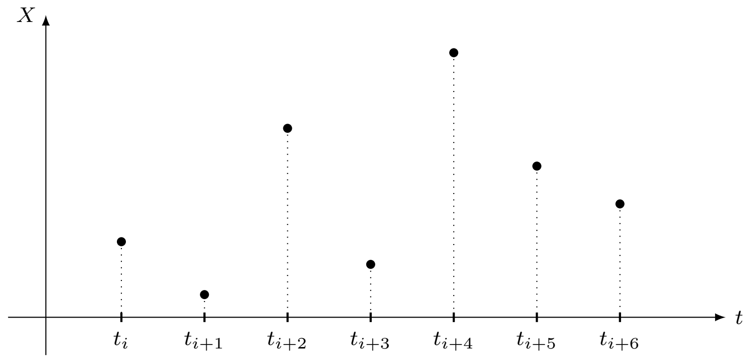


Figure II.1: A discrete function

Definition II.4 (Discretization of functions)**.** *Let $x \in \mathcal{F}([a,b],\mathbb{R})$, a discretization of x is a discrete function X such that each X_i can be computed using the values $\{x_i = x(t_i)\}_{i=0,\dots,N}$ such that*

$$|X_i - x(t_i)| \leq Ch^\alpha$$

where $C > 0$ and $\alpha > 0$ are two constant, independent of h, and α independent of i.

A *natural* or *canonical discretization* is of course given by the restriction of x to $\mathbb{T}$.

Definition II.5 (Canonical discretization)**.** *Let $I \subset [a,b]$ be an interval and $I_\mathbb{T} = \mathbb{T} \cap I$. We denote by $\pi : \mathcal{F}(I,\mathbb{R}) \to \mathcal{F}(I_\mathbb{T},\mathbb{R})$ the mapping defined by the restriction of a given function $x \in \mathcal{F}(I,\mathbb{R})$ to $I_\mathbb{T}$, i.e., $\pi(x) = X$ where $X_i = x(t_i)$, $t_i \in I_\mathbb{T}$.*

The following section introduces two *lift map* from $\mathcal{F}(\mathbb{T},\mathbb{R})$ in various finite vector subspaces of $\mathscr{C}([a,b],\mathbb{R})$.

II.4.2 Interpolation mappings and lifting of discrete functions

In order to construct discrete analogue of continuous mappings, we have to relate the set of discrete functions to some finite vector spaces of $\mathscr{C}([a,b],\mathbb{R})$.

II.4.2.1 Interpolation mappings

The notion of *interpolation* can be formally defined as follows:

Definition II.6 (Interpolation map)**.** *A map $\kappa : \mathcal{F}(\mathbb{T},\mathbb{R}) \to \mathscr{C}([a,b],\mathbb{R})$ satisfying $\pi \circ \kappa = \mathrm{Id}$, where Id is the identity of $\mathcal{F}(\mathbb{T},\mathbb{R})$ is called an interpolation map.*

The name *interpolation* comes from the last condition. One can naturally extend a given interpolation mapping for discrete functions with values in $\mathbb{R}^d$ by posing for $X \in \mathcal{F}(\mathbb{T},\mathbb{R}^d)$, $X = (X^1,\dots,X^d)$, $X^i \in \mathcal{F}(\mathbb{T},\mathbb{R})$, $\kappa(X) = \big(\kappa(X^1),\dots,\kappa(X^d)\big)$.

Three classical interpolation mappings will be used in the following (see [35, Chap. II, §.1]).

Definition II.7 ($\mathscr{P}_0^\pm$-interpolation)**.** *We denote by $\kappa_0^\pm : \mathcal{F}(\mathbb{T},\mathbb{R}) \to \mathscr{P}_0^\pm([a,b],\mathbb{R})$ the map defined for all $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ by*

$$\kappa_0^+(X) = \sum_{i=0}^{N-1} X_i \mathbb{1}_{[t_i,t_{i+1}[} \quad \left(\text{resp. } \kappa_0^-(X) = \sum_{i=1}^{N} X_i \mathbb{1}_{]t_{i-1},t_i]}\right). \tag{II.4.1}$$

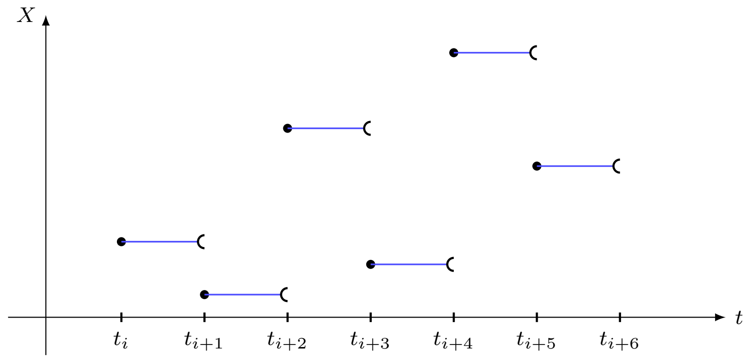


Figure II.2: $\mathscr{P}_0^+$-interpolation mapping

We sometimes need more regularity.

Definition II.8 ($\mathscr{P}_1$-interpolation)**.** *We denote by* $\kappa_1 : \mathcal{F}(\mathbb{T}, \mathbb{R}) \to \mathscr{P}_1([a, b], \mathbb{R})$ *the map defined for all* $X \in \mathcal{F}(\mathbb{T}, \mathbb{R})$ *by*

$$\kappa_1(X)(t) = P_{1,\ell}^{X_i, X_{i+1}}(t) \quad \textit{for} \quad t_i \leq t \leq t_{i+1},\ i = 0, \ldots, N-1 \tag{II.4.2}$$

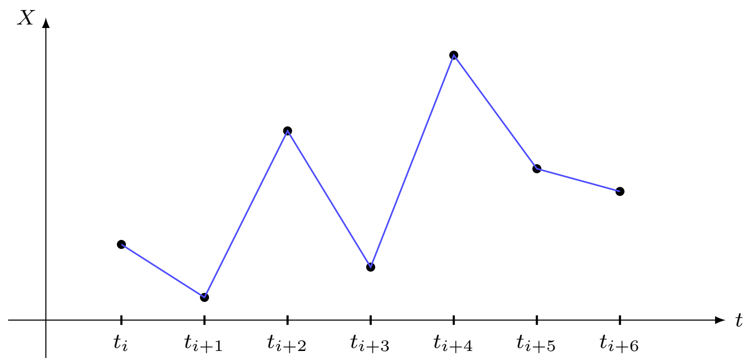


Figure II.3: $\mathscr{P}_1$-interpolation mapping

This map is of course natural and is classical in numerical analysis. The main problem is that in this case, the derivative on $\mathbb{T}$ is not defined but only left and right derivatives. This point will induce several complications in the connection between the various discrete operators.

II.4.2.2 Properties of interpolation mappings

The interpolation mappings satisfy some additional interesting properties:

Lemma II.1. *The restriction of* $\kappa_0^+ \circ \pi$ *and* $\kappa_0^- \circ \pi$ *to* $\mathscr{P}_0^+([a, b], \mathbb{R})$ *and* $\mathscr{P}_0^-([a, b], \mathbb{R})$ *respectively is the identity.*

II.4.3 Order of an Embedding of a linear operator

Let O be a linear operator defined on its domain $\text{Dom}(O) \subset \mathcal{F}([a, b], \mathbb{R}^d)$ and takes its values in $\text{Im}(O) \subset \mathcal{F}([a, b], \mathbb{R}^d)$. We define:

Definition II.9. *Let* $\text{Dom}(O_h) \subset \mathcal{F}(\mathbb{T}, \mathbb{R}^d)$ *and* $\text{Im}(O_h) \subset \mathcal{F}(\mathbb{T}, \mathbb{R}^d)$. *Let* κ_O *be an interpolation map from* $\text{Dom}(O_h)$ *to* $\text{Dom}(O)$. *The discrete embedding* O_h *of* O *defined by* $O_h = \pi \circ O \circ \kappa_O$ *is of order* $r > 0$ *if*

$$\|\pi\left(O(\kappa(X))\right) - O_h(X)\|_{\mathcal{F}([a,b],\mathbb{R}^d)} \leq Ch^r, \tag{II.4.3}$$

for all $X \in \text{Dom}(O_h)$ *and where* C *is a constant independent of* h.

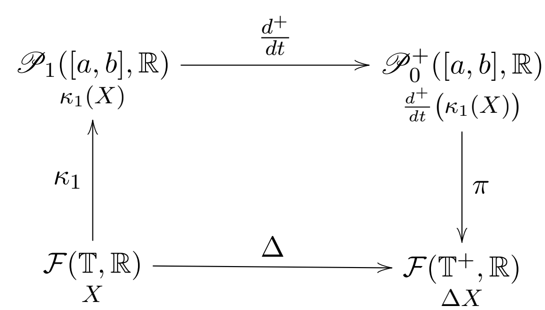
II.5 Discrete derivatives

The interpolation mapping κ_1 is not differentiable but we can always define a left and right derivative in the points t_i. As a consequence, we can define Δ and ∇ the two discrete operators corresponding to the left and right derivative denoted by $\frac{d^+}{dt}$ and $\frac{d^-}{dt}$ as follows:

Definition II.10. *The forward (resp. backward) discrete derivative* Δ *(resp.* ∇*) is defined by*

$$\Delta = \pi \circ \frac{d^+}{dt} \circ \kappa_1 \quad \left(\textit{resp.} \quad \nabla = \pi \circ \frac{d^-}{dt} \circ \kappa_1 \right). \tag{II.5.1}$$

This definition corresponds to the following commutative diagram



We deduce easily from the previous definition that:

Lemma II.2. *The forward (resp. backward) discrete derivative* Δ *(resp.* ∇*) takes its values in* $\mathcal{F}(\mathbb{T}^+, \mathbb{R})$ *(resp.* $\mathcal{F}(\mathbb{T}^-, \mathbb{R})$*). Moreover, the forward and the backward discrete derivatives are surjective.*

Proof. We have $\frac{d^+}{dt}(\mathscr{P}_1([a,b],\mathbb{R}) = \mathscr{P}_0^+([a,b],\mathbb{R})$ and $\frac{d^-}{dt}(\mathscr{P}_1([a,b],\mathbb{R}) = \mathscr{P}_0^-([a,b],\mathbb{R})$. We deduce that

$$\pi(\mathscr{P}_0^+([a,b],\mathbb{R})) = \mathcal{F}([a,b[\cap\mathbb{T},\mathbb{R}) = \mathcal{F}(\mathbb{T}^+,\mathbb{R}),$$

and

$$\pi(\mathscr{P}_0^-([a,b],\mathbb{R})) = \mathcal{F}(]a,b]\cap\mathbb{T},\mathbb{R}) = \mathcal{F}(\mathbb{T}^-,\mathbb{R}).$$

□

The following figure illustrates the definition of the forward discrete derivative.

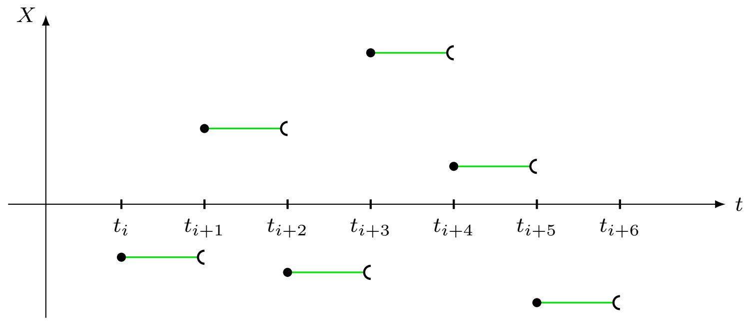


Figure II.4: Forward discrete derivative

II.5.1 Explicit form

It is easy to obtain an explicit form for these two discrete derivatives:

Lemma II.3. *Let $X \in \mathcal{F}(\mathbb{T}, \mathbb{R})$, we have*

$$(\Delta F)_i = \frac{X_{i+1} - X_i}{h} \quad \textit{for} \quad i = 0, \ldots, N-1,$$

and

$$(\nabla F)_i = \frac{X_1 - X_{i-1}}{h} \quad \textit{for} \quad i = 1, \ldots, N.$$

We recover the classical forward and backward derivatives used in numerical analysis.

II.5.2 Properties of discrete derivatives

II.5.2.1 Duality

The duality between $\mathcal{F}(\mathbb{T}^+, \mathbb{R})$ and $\mathcal{F}(\mathbb{T}^-, \mathbb{R})$ can be used to obtain a duality between the Δ and ∇ derivative. Precisely, we have:

Lemma II.4. *Let $X \in \mathcal{F}(\mathbb{T}, \mathbb{R})$. We have*

$$\begin{aligned} \nabla(\sigma(X)) &= \Delta F \quad \textit{over} \quad \mathbb{T}^+ \\ \Delta(\rho(X)) &= \nabla F \quad \textit{over} \quad \mathbb{T}^- \end{aligned}$$

II.5.2.2 Kernel of discrete derivatives

Lemma II.5. *Let $X \in \mathcal{F}(\mathbb{T}, \mathbb{R})$. We have $\Delta F = 0$ (resp. $\nabla F = 0$) if and only if $X = \mathbb{F}_0$, where $\mathbb{F}_0$ is defined by $\pi(X_0 \mathbb{1}_{\mathbb{T}})$. We will also say that X is constant.*

The previous Lemma can be used for example to characterize *discrete* first integrals of classical numerical scheme (see for example [15, Theorem 12, p.885] and compare with [45, Theorem 6.7, p.197]).

II.5.2.3 Discrete Leibniz formula

An important algebraic property of the classical derivative is the Leibniz formula. The following theorem gives the discrete version of this formula which mimics exactly the continuous one.

Theorem II.1 (Discrete Leibniz rule). *Let $X, Y \in \mathcal{F}(\mathbb{T}, \mathbb{R})$. We have*

$$\begin{aligned} \Delta(X \times Y) &= (\Delta X) \times Y + \sigma(X) \times (\Delta Y) \quad \textit{over} \quad \mathbb{T}^+, \\ \nabla(X \times Y) &= (\nabla X) \times Y + \rho(X) \times (\Delta Y) \quad \textit{over} \quad \mathbb{T}^-. \end{aligned} \tag{II.5.2}$$

It is interesting to notice that despite the constant use of the previous formula in many computations concerning numerical scheme, very few exhibits the fact that this is a discrete Leibniz formula. This is due in part to the fact that most of the computations are usually made directly using summations and not in an abstract way. This phenomenon is particularly visible in the derivation of *variational integrators* (see [74]) in the context of the *discrete calculus of variations*.

II.5.3 Comparison with the time-scale calculus

The formulas obtained in Lemma II.4 and the discrete Leibniz rule also appear in the time-scale calculus (see [11], [14]). From two different approach of the discrete calculus, this coincidence is related to the fact that, we have considered here as an illustration of our framework, interpolation mapping of order one. It gives finite difference of order one, also called, Euler forward and backward methods which is exactly the methods used by the time-scale calculus at the discrete level.

II.5.4 Extension of the discrete derivatives over $\mathbb{T}$

It is possible to define ρ, σ and the discrete derivatives over $\mathbb{T}$ by choosing an ad-hoc extension of $\kappa_1(X)$ outside $[a, b]$. Indeed, we consider the continuous extension of $\kappa_1(X)$ defined by X_0 for all $t \leq a$ and by X_N for all $t \geq b$. In that case, we extend the definition of ρ (resp. σ) at t_0 (resp. at t_N) as

$$\rho(X)(t_0) = X_0 \quad \text{and} \quad \sigma(X)(t_N) = X_N.$$

Then, we obtain by definition of the discrete derivatives

$$(\nabla F)_0 = (\Delta F)_N = 0.$$

II.6 Discrete antiderivatives

Using the same procedure as for discrete derivatives, we define discrete antiderivatives.

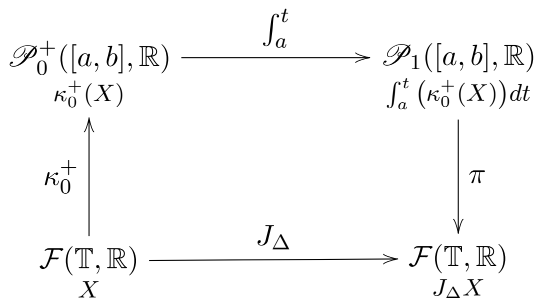
II.6.1 Discrete antiderivatives

Using the lift mappings we can easily define an antiderivative:

Definition II.11 (Discrete antiderivative). *The discrete Δ (resp. ∇) antiderivative denoted by J_Δ (resp. J_∇) is defined by*

$$J_\Delta = \pi \circ \int_a^t \circ \kappa_0^+ \quad \left(\textit{resp.} \quad J_\nabla = \pi \circ \int_a^t \circ \kappa_0^- \right). \tag{II.6.1}$$

this definition for J_Δ corresponds to the following diagram



An analogous diagram is obtained for J_∇.

II.6.2 Explicit form

An explicit formula for J_Δ (resp. J_∇) is easily obtained:

Lemma II.6. *Let $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$. We have*

$$\big[J_\Delta(X)\big]_i = \sum_{k=0}^{i-1}(t_{k+1}-t_k)\,X_k \qquad \left(\textit{resp.} \quad \big[J_\nabla(X)\big]_i = \sum_{k=1}^{i}(t_k-t_{k-1})\,X_k\right)$$

for $i = 0, \ldots, N$.

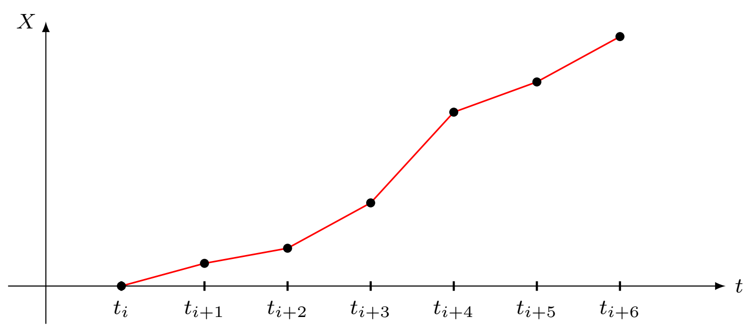


Figure II.5: Discrete Δ-integral

II.6.3 Properties of discrete antiderivatives

Duality. The duality between the Δ and ∇ derivatives induces a corresponding phenomenon for the Δ and ∇ antiderivatives.

Proposition II.1. *For all $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ we have $J_\Delta \circ \sigma(X) = J_\nabla(X)$ and $J_\nabla \circ \rho(X) = J_\Delta(X)$.*

Discrete integration by parts. As in the continuous case, the discrete Leibniz formula for discrete derivatives induces a *discrete integration by parts* formula for discrete antiderivatives.

Theorem II.2. *Let $X, Y \in \mathcal{F}(\mathbb{T},\mathbb{R})$, we have*

$$\begin{aligned} \big[J_\Delta(X \times \Delta(Y))\big]_N &= X_N Y_N - X_0 Y_0 - \big[J_\Delta(\Delta X \times \sigma(Y))\big]_N, \\ \big[J_\nabla(X \times \nabla(Y))\big]_N &= X_N Y_N - X_0 Y_0 - \big[J_\nabla(\nabla X \times \rho(Y))\big]_N. \end{aligned} \tag{II.6.2}$$

Proof. We only make the proof for J_Δ, the case for J_∇ is similar. Using the discrete Leibniz formula for Δ, we have

$$J_\Delta(X \times \Delta Y) = J_\Delta\left(\Delta(X \times Y) - \Delta X \times \sigma(Y)\right).$$

Using Theorem II.4 we obtain

$$\left(J_\Delta \circ \Delta(X \times Y)\right)_N = X_N Y_N - X_0 Y_0,$$

which concludes the proof. □

This form induces shifts in the discrete integral in the right hand side but using Lemma II.4, we can obtain an alternative form:

Theorem II.3 (Discrete integration by parts). *Let $X, Y \in \mathcal{F}(\mathbb{T}, \mathbb{R})$, we have*

$$\begin{aligned} \left[J_\Delta(X \times \Delta Y)\right]_N &= \rho(X)_N Y_N - \rho(F)_0 Y_0 - \left[J_\Delta(\nabla F \times Y)\right]_N, \\ \left[J_\nabla(X \times \nabla Y)\right]_N &= \sigma(X)_N Y_N - \sigma(F)_0 G_0 - \left[J_\nabla(\Delta X \times Y)\right]_N. \end{aligned} \tag{II.6.3}$$

The previous result is never stated as above in classical Textbooks about finite differences. This is due to the fact that the operators are not usually used to write such a formula but directly on summations formula only speaking of rearranging the terms of the sum (see [74, p.363] for a typical example).

Scalar product. Let $x, y \in \mathscr{C}([a,b], \mathbb{R}^d)$. The classical scalar product on L^2 functions denoted by $\langle \cdot, \cdot \rangle_{L^2}$ is defined by $\langle f, g \rangle_{L^2} = \int_a^b \langle X(t), Y(t) \rangle \, dt$. Using the *discrete product* and the lift mappings, we can transport the L^2 scalar product over discrete functions.

Definition II.12 (Discrete scalar product). *We call forward (resp. backward) discrete scalar product the bilinear mapping defined for all $X, Y \in \mathcal{F}(\mathbb{T}, \mathbb{R}^d)$ by*

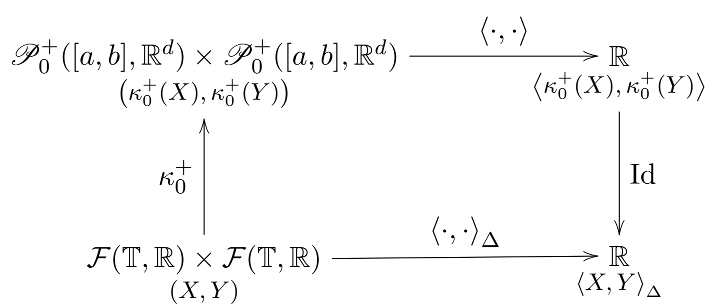
$$\langle X, Y \rangle_+ = J_\Delta\left[\, \langle X, Y \rangle_\times \,\right]_N \quad \left(\textit{resp.} \quad \langle X, Y \rangle_- = J_\nabla\left[\, \langle X, Y \rangle_\times \,\right]_N \right). \tag{II.6.4}$$

Remark II.1. *The discrete forward (resp. backward) discrete scalar product is degenerate. Indeed, the equation $\langle X, X \rangle_+ = 0$ (resp. $\langle X, X \rangle_- = 0$) induces only*

$$X_i = 0 \quad \textit{for} \quad i = 0, \ldots, N-1, \quad (\textit{resp.} \quad X_i = 0 \quad \textit{for} \quad i = 1, \ldots, N). \tag{II.6.5}$$

Using the definition of discrete antiderivatives, we prove the following lemma:

Lemma II.7. *The following diagram commutes*



The same results occur with ∇ instead of Δ and κ_0^- instead of κ_0^+.

In other word, the discrete scalar product is just the usual L^2 scalar product with L^2 functions replaced by discrete functions, the usual multiplication by its discrete analogue and the classical antiderivative by its discrete pendant. This phenomenon is in fact general. We will formalize this property in the next part using discrete embedding formalisms.

II.7 Discrete classical results

In this section, we derive two discrete analogue of classical results in Analysis: the fundamental theorem of differential calculus and the Dubois-Reymond lemma.

II.7.1 Discrete fundamental theorem of differential calculus

As we are looking for the transfer of algebro-analytic properties of derivatives and antideratives a natural question is up to which extent the *fundamental theorem of differential calculus* (see [44, Theorem 6.13, p.239]) is preserved ? A very nice feature of our derivation is that this theorem is the following discrete analogue of this result:

Theorem II.4 (Fundamental theorem of the discrete differential calculus)**.** *For all $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ we have*

$$\begin{aligned} J_\Delta \circ \Delta(X) &= X - \mathbb{X}_0 \quad \textit{and} \quad \Delta \circ J_\Delta X = X, \\ J_\nabla \circ \nabla(X) &= X - \mathbb{X}_0 \quad \textit{and} \quad \nabla \circ J_\nabla X = X. \end{aligned} \tag{II.7.1}$$

Proof. We make the proof only for Δ and J_Δ. The proof for ∇ and J_∇ being similar.

As $\frac{d^+}{dt}\circ\kappa_1(X) \in \mathscr{P}_0^+([a,b],\mathbb{R})$, we deduce that $\kappa_0^+\circ\pi(\frac{d^+}{dt}\circ\kappa_1(X)) = \frac{d^+}{dt}\circ\kappa_1(X)$. As a consequence, we obtain

$$\begin{aligned} J_\Delta \circ \Delta(X) &= \pi \circ \int_a^t \circ\, \kappa_0^+ \circ \pi \circ \frac{d^+}{dt} \circ \kappa_1(X) \\ &= \pi \circ \int_a^t \frac{d^+}{dt}(\kappa_1(X)) \\ &= \pi\left(\kappa_1(X)(t) - \kappa_1(X)(a)\right) = X - \mathbb{X}_0. \end{aligned}$$

Second we remark that over $\mathscr{P}_1([a,b],\mathbb{R})$ we have $\kappa_1 \circ \pi = \mathrm{Id}$. As $\int_a^t \kappa_0^+(X) \in \mathscr{P}_1([a,b],\mathbb{R})$ then by definition we have

$$\begin{aligned} \Delta \circ J_\Delta(X) &= \pi \circ \frac{d^+}{dt} \circ \kappa_1 \circ J_\Delta(X) \\ &= \pi \circ \frac{d^+}{dt} \circ \kappa_1 \circ \pi \circ \int_a^t \kappa_0^+(X) \\ &= \pi\left(\frac{d^+}{dt} \circ \int_a^t \kappa_0^+(X)\right) = \pi\left(\kappa_0^+(X)(t)\right) = X. \end{aligned}$$

This conclude the proof. □

II.7.2 Discrete Dubois-Reymond lemma

The discrete version of the Dubois-Reymond lemma is valid. We first introduce the set $\mathcal{F}_0(\mathbb{T},\mathbb{R}) \subset \mathcal{F}(\mathbb{T},\mathbb{R})$ defined by

$$\mathcal{F}_0(\mathbb{T},\mathbb{R}) = \{Y \in \mathcal{F}(\mathbb{T},\mathbb{R}),\ Y_0 = Y_N = 0\}. \tag{II.7.2}$$

Lemma II.8. *Let $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ such that $\left[J_\Delta(X \times Y)\right]_N = 0$ for all $Y \in \mathcal{F}_0(\mathbb{T},\mathbb{R})$ then $X_i = 0$ for $i = 1,\ldots,\ N-1$.*

Proof. Let $Y \in \mathcal{F}_0(\mathbb{T},\mathbb{R})$. We choose Y such that $Y_i = X_i$ for $1 \leq i \leq N-1$. Hence, we obtain

$$\left[J_\Delta(X \times Y)\right]_N = \sum_{i=1}^{N-1} h\, X_i^2 = 0.$$

We deduce $X_i = 0$ for $i = 1,\ldots,N-1$. This concludes the proof. □

A stronger result can be derived when X is replaced by ΔX:

Lemma II.9. *Let $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ such that $\left[J_\Delta(\Delta X \times Y)\right]_N = 0$ for all $Y \in \mathcal{F}(\mathbb{T}^+,\mathbb{R})$ then $\Delta X = 0$.*

Proof. The proof follows from a simple computation. As

$$\left[J_\Delta(\Delta X \times Y)\right]_N = \sum_{i=0}^{N-1} h\, \frac{X_{i+1} - X_i}{h} Y_i,$$

we obtain, by taking $Y = \Delta X \in \mathcal{F}(\mathbb{T}^+,\mathbb{R})$ that $X_{i+1} - X_i = 0$ for all $i = 0,\ldots,N-1$, so that $X = \mathbb{X}_0$. As a consequence, X is a constant for Δ which concludes the proof. □

Similar results can be obtained for the ∇-derivative.

II.8 Discrete embedding formalisms

We use the previous abstract approach to finite differences in order to provide a formal definition of a discrete analogue for differential equations, functional and other objects which can be defined using integrals, derivatives and functions. We first define a general abstract procedure and give three natural discrete generalization of a differential equation. This point of view allows us to more precisely determine on which assumptions a discretization can be constructed.

II.8.1 Abstract discrete embedding

In this section, we consider a general *formal functional* made of symbols d/dt and $\int$ acting on a given set of functions x, y, etc. We denote such a formal functional by

$$F\left(t, x, y; d/dt, \int\right), \tag{II.8.1}$$

as long as the expression defined by X is well defined. A *formal relation* will be the data of a formal functional satisfying

$$F\left(t, x, y; d/dt, \int\right) = 0. \tag{II.8.2}$$

A classical formal relation is given by a first order differential equation

$$\frac{dx}{dt} - f(t,x) = 0.$$

We can now define what is a discrete embedding of an abstract functional or relation.

Definition II.13 (Abstract discrete embedding)**.** *Let X be a formal functional defined by* (II.8.1)*. The discrete embedding of X denoted by $X_{\mathcal{E}}$ is defined for all $X \in \mathcal{E}$ by*

$$F_{\mathcal{E}}(T, X) = F\left(T, X; (d/dt)_{\mathcal{E}}, \int_{\mathcal{E}}\right), \tag{II.8.3}$$

where $\mathcal{E}$, $(d/dt)_{\mathcal{E}}$ and $\int_{\mathcal{E}}$ are the discrete functional space, the discrete derivative and discrete antiderivative which are fixed.

Remark II.2. *The previous procedure deals with the minimal objects used to write a given differential relation. In many situations as for example geometry, a differential relation put in evidence some particular operators, like the Laplacian, from which a differential equation is constructed. In such a context, one can be conducted to define a discrete embedding directly focusing on the given operator and its algebraic or geometric properties. The discrete embedding then follows the same lines as in Definition II.13 but the "functorial" property that we are looking for is destroyed. We refer to [27] for more details.*

II.9 Application to ordinary differential equations: the three forms

In this section, we apply the previous formalism for ordinary differential equations. Using different representations of a given differential equation (differential or integral form, variational), we obtain discrete analogues which do not always give the same object. We have then multiplicity of *discrete realizations* of a given differential equations. This problem is in fact relevant in all embedding formalism and leads to the *coherence problem* which consist in finding the conditions under which such representations coincide.

II.9.1 Discrete differential embedding

Let $x \in \mathbb{R}^d$, we consider the ordinary differential equation

$$\frac{dx}{dt} = f(t, x). \tag{II.9.1}$$

Using the finite differences embedding, the discrete Δ-version of this equation is

$$\Delta X = f(T, X),\ X \in \mathcal{F}(\mathbb{T}, \mathbb{R}^d),\ T \in \mathcal{F}(\mathbb{T}, \mathbb{R}) \tag{II.9.2}$$

As ΔX is defined on $\mathbb{T}^+$, we obtain for each $i = 0, \dots, N-1$

$$\frac{X_{i+1} - X_i}{h} = f(T_i, X_i), \tag{II.9.3}$$

where $T_i = a + i(b-a)/h$. Also, we have the discrete ∇-version of this equation which is

$$\nabla X = f(T, X)\ , X \in \mathcal{F}(\mathbb{T}, \mathbb{R}^d), T \in \mathcal{F}(\mathbb{T}, \mathbb{R}). \tag{II.9.4}$$

As ∇X is defined on $\mathbb{T}^-$, we obtain for each $i = 1, \dots, N$

$$\frac{X_i - X_{i-1}}{h} = f(T_i, X_i), \tag{II.9.5}$$

II.9.2 Discrete integral embedding

The integral formulation of the previous ordinary differential equation is given by

$$x(t) \ = x(0) + \int_0^t f(s, x(s)) ds. \tag{II.9.6}$$

The forward discrete embedding of this equation is then given by

$$X = \mathbb{X}_0 + J_\Delta(f(T, X)), \tag{II.9.7}$$

which we call the *forward discrete integral embedding.* By definition, this equation is equivalent to

$$X_i = X_0 + h \sum_{k=0}^{i-1} f(T_k, X_k), \quad i = 1, \dots, N.$$

As a consequence, we obtain $X_{i+1} - X_i = hf(T_i, X_i)$, $i = 0, \dots, N-1$, which is the classical *one-step forward Euler scheme* in Numerical Analysis (see [35, V, §.2.3]).

In the same way one can obtain the *backward discrete integral embedding* of this equation which is then given by

$$X = \mathbb{X}_0 + J_\nabla(f(T, X)). \tag{II.9.8}$$

By definition, this equation is equivalent to

$$X_i = X_0 + h \sum_{k=0}^{i-1} f(T_k, X_k), \quad i = 1 \dots, N.$$

Easy computations lead to $X_i - X_{i-1} = hf(T_i, X_i)$, $i = 0, \dots, N-1$, which is the classical *one step backward Euler scheme* in Numerical Analysis.

Remark II.3. *The finite differences integral embedding of the equation coincides with the differential case. As a consequence, we see that in this simple case, we have coherence between the two discrete versions of the equation.*

II.9.3 Discrete variational embedding

In this section, we define the discrete variational embedding of a second order differential equation which is Lagrangian. Our derivation is compared with the classical work of J. E. Marsden and M. West [74] (see also [45]) about the discrete calculus of variation and variational integrators, in order to explain the interest of our abstract framework.

II.9.3.1 Discrete Lagrangian functional

The discrete calculus of variations is defined over discrete Lagrangian functional which are obtained using the discrete embedding that we have fixed.

Definition II.14. *Let L be an admissible Lagrangian function and $\mathcal{L}$ the associated functional. The discrete forward Lagrangian functional $\mathcal{L}_\Delta$ associated to L with the Δ-integral is defined by*

$$\mathcal{L}_\Delta(X) = \left[J_\Delta(L(T, X, \Delta X)) \right]_N. \tag{II.9.9}$$

The previous form is completely fixed once one has given a discrete embedding formalism. We clearly see the relation between the classical function and the discrete one.

II.9.3.2 Comparison with Marsden-West definition

In [74, p.363 and §.1.3 p.370-371], the authors define discrete Lagrangian system for which they do not preserve the classical form of the continuous functional, i.e., their form does not put in evidence the integral and the differential structure of the continuous case. They introduce a discrete Lagrangian given by

$$L_d(h, X_i, X_{i+1}) = hL(t_i, X_i, \Delta(X)_i), \tag{II.9.10}$$

which gives the discrete functional

$$\mathcal{L}_h(X) = \sum_{i=0}^{N-1} L_d(h, X_i, X_{i+1}). \tag{II.9.11}$$

The algebraic structure of the classical functional is then destroyed in a sense that we can not obtain the discrete analogue of the integral and the derivative of the continuous Lagrangian functional. This point has an important consequence in the derivation of the discrete Euler-Lagrange equation in [74, p.363 and §.1.3, Theorem 1.3.1, p.371]. Indeed, as we will see, the form obtained in [74] does not put in evidence the complete analogy between the discrete Euler-Lagrange equation and the classical one.

The same remark applies to the presentation made by E. Hairer, C. Lubich and G. Wanner in the book *Geometric Numerical Integration* (see [45, Formula (6.5) and (6.6) p.192-195]).

II.9.4 Discrete calculus of variations and discrete Euler-Lagrange equations

An element of $\mathcal{F}_0(\mathbb{T},\mathbb{R})$ is called a *discrete variation.* As $\mathcal{F}(\mathbb{T},\mathbb{R})$ is a linear space, we can define the Fréchet derivative of $\mathcal{L}_\Delta$ (resp. $\mathcal{L}_\nabla$) along a given direction $H \in \mathcal{F}(\mathbb{T},\mathbb{R})$ and denoted by

$$D\mathcal{L}_\Delta(X)(H) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(\mathcal{L}_\Delta(\mathrm{X} + \epsilon H) - \mathcal{L}_\Delta(X)). \tag{II.9.12}$$

The corresponding notion of critical points is given by:

Definition II.15. *A discrete critical point $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ verify $D\mathcal{L}_\Delta(X)(H) = 0$ for all $H \in \mathcal{F}_0(\mathbb{T},\mathbb{R})$.*

We obtain the following *discrete Euler-Lagrange equation*:

Theorem II.5 (Discrete Euler-Lagrange equation)**.** *Let L be an admissible Lagrangian function. A discrete function $X \in \mathcal{F}(\mathbb{T},\mathbb{R})$ is a critical point of the discrete Δ Lagrangian functional associated to L if and only if it satisfies*

$$\nabla\left(\frac{\partial L}{\partial v}(T, X, \Delta X)\right) = \frac{\partial L}{\partial x}(T, X, \Delta X), \quad \textit{over} \quad \mathbb{T}^{\pm}. \tag{II.9.13}$$

The proof can be done in different ways. However, we want to keep a proof which is similar to the classical proof of the continuous Euler-Lagrange equation as exposed for example in [42, Theorem 1, p.15].

Proof. Using a Taylor expansion of L, we obtain

$$D\mathcal{L}_\Delta(X)(H) = \left[J_\Delta\left(\frac{\partial L}{\partial v}(T, X, \Delta X) \times \Delta H + \frac{\partial L}{\partial x}(T, X, \Delta X) \times H\right)\right]_N. \tag{II.9.14}$$

As $H \in \mathcal{F}_0(\mathbb{T},\mathbb{R})$, we have using the discrete integration by parts formula

$$D\mathcal{L}_\Delta(X)(H) = \left[J_\Delta \left(-\nabla \left(\frac{\partial L}{\partial v}(T,X,\Delta X) \right) \times H + \frac{\partial L}{\partial x}(T,X,\Delta X) \times H \right) \right]_N . \tag{II.9.15}$$

Using the discrete Dubois-Reymond lemma, we deduce

$$-\nabla \left[\frac{\partial L}{\partial v}(T,X,\Delta X) \right] + \frac{\partial L}{\partial x}(T,X,\Delta X) = 0 \text{ over } \mathbb{T}^{\pm}. \tag{II.9.16}$$

This concludes the proof. □

The previous formulation makes clear the relation between the usual Euler-Lagrange equation and the discrete one. In particular, one see the duality between the two operators Δ and ∇ which induces a natural mixing between the two discrete derivatives which is hide in the continuous case.

II.9.5 Comparison with Marsden-West definition

One can compare this writing of the discrete Euler-Lagrange equation with the one obtained in [74, p.363]. Using the form (II.9.11), they have

$$D_2 L_d(X_{i-1}, X_i, h) + D_1 L_d(X_i, X_{i+1}, h) = 0, \tag{II.9.17}$$

for $i = 1, \dots, N-1$. The usual form of the Euler-Lagrange equation is completely lost and by the way the analogy between the two objects.

The same remark applies to the presentation made by E. Hairer, C. Lubich and G. Wanner in the book *Geometric Numerical Integration* (see [45, Formula (6.7) p.192-195]).

Chapter III

High-order Time-scale Calculus and Galerkin Variational Integrators

In this chapter, we extend the classical time-scale calculus in order to obtain a high-order approximation of classical continuous operators for discrete time-scales. As an application, we derive Galerkin variational integrators and compare our formulation with the usual Marsden's approach.

*This Chapter is based on the preprint "*High-order time-scale calculus and Galerkin variational integrators*" with J. Cresson and A. Szafrańska from the Gdansk University of Technology, Poland.*

III.1 Introduction

Finite differences or time scales calculus over discrete time scales lead to approximation of derivatives and integrals of order one. For the purpose of numerical analysis, such a limitation is very strong. As a consequence, people have developed numerical methods of high order. This is the case of *Galerkin variational integrators* or *higher order variational integrators* as initiated by J.E. Marsden et M. West [74] and generalized by Leok [48], [68], [69]. The analysis of this method was treated for example by S. Ober-Blöbaum and Saak [82], Hal [46], Hall and Leok [47].

However, the same problems as for the classical presentation of discrete variational integrators by J.E. Marsden and M. West as explained in Chapter II appear, i.e. namely that the structure of the high-order calculus of variations as well as the formulation of the high-order Euler-Lagrange equation does not put in evidence the explicit connection between these equations. This is due as usual to the fact that no high order discrete derivative or antiderivative is defined over the discrete functional set used to derive such integrators.

Organization of the chapter. In Section III.3 and III.4, we give definitions concerning control time scale, control function and High-order interpolation. In Sections III.5 and III.6, we briefly present some properties of the usual discrete integrals, then we define a discrete operator extending the classical Δ anti-derivative on time scales using the classical notion of quadrature. Section III.7 is devoted to replace the classical Galerkin approach introduced in [74] in the high order discrete embedding framework.

III.2 Notations

Let $\mathbb{T} = \{t_i = a + ih,\ i = 0, \ldots, N\}$ be the usual time scale on $[a, b]$ where $N \in \mathbb{N}^*$ and $h > 0$. We will use here the notation $\mathbb{T}^- = \mathbb{T} \setminus \{a\}$, $\mathbb{T}^+ = \mathbb{T} \setminus \{b\}$, $\mathbb{T}^\pm = \mathbb{T}^+ \cap \mathbb{T}^-$ and $I_i = [t_i, t_{i+1}]$, $I_i^+ = [t_i, t_{i+1}[$, $I_i^- =]t_i, t_{i+1}]$ and $I_i^\pm =]t_i, t_{i+1}[$, subintervals of $[a, b]$.

We denote by $\sigma : \mathrm{C}(\mathbb{T}^+, \mathbb{R}) \to \mathrm{C}(\mathbb{T}^-, \mathbb{R})$ (resp. $\rho : \mathrm{C}(\mathbb{T}^-, \mathbb{R}) \to \mathrm{C}(\mathbb{T}^+, \mathbb{R})$) the map defined by

$$\sigma(F(t_i)) = F(t_{i+1}),\ i = 0, \ldots, N-1 \quad (\text{resp. } \rho(F(t_i)) = F(t_{i-1}),\ i = 1, \ldots, N).$$

Let $\mathbb{T}_\eta$ be a time scale defined on $[0, 1]$ by

$$\mathbb{T}_\eta = \{\eta_i, i = 0, \ldots, m\} \quad \text{with} \quad \eta_0 = 0,\ \eta_i < \eta_{i+1},\ \eta_m = 1. \tag{III.2.1}$$

In the following we denote by $\mathscr{P}_m([a, b], \mathbb{R})$ with $m \in \mathbb{N}$, the set of continuous functions that are polynomials of degree $\leq m$.

Definition III.1. *A basis of $\mathscr{P}_m([0, 1], \mathbb{R})$ is given by the $\mathbb{T}_\eta$-Lagrange polynomials defined for all $y \in [0, 1]$ by*

$$\ell_j(y) = \prod_{\substack{j \neq i \\ i=0}}^{m} \frac{(y - \eta_i)}{(\eta_j - \eta_i)}, \quad j = 0, \ldots, m.$$

We want to define a polynomial interpolation of $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ over each $I_i,\ i = 0, \ldots, N-1$.

III.3 Interpolation map of degree m

Let $\mathscr{P}_m([a, b], \mathbb{R})$ be the set of piecewise polynomial functions of degree m over each $I_i,\ i = 0, \ldots, N-1$. In order to characterize an element of $\mathscr{P}_m([0, 1], \mathbb{R})$, we need in each I_i the data $m+1$ points of interpolation, this can be done by introducing a time scale $\mathbb{T}_\eta$ containing $m+1$ points over $[0, 1]$ which induces a time scale over each I_i and a control function X_c defined over these time scales.

Definition III.2. *A control of order m, $m \geq 1$, over $\mathbb{T}$ denoted by $(\mathbb{T}_\eta, X_c)$ is a data of*

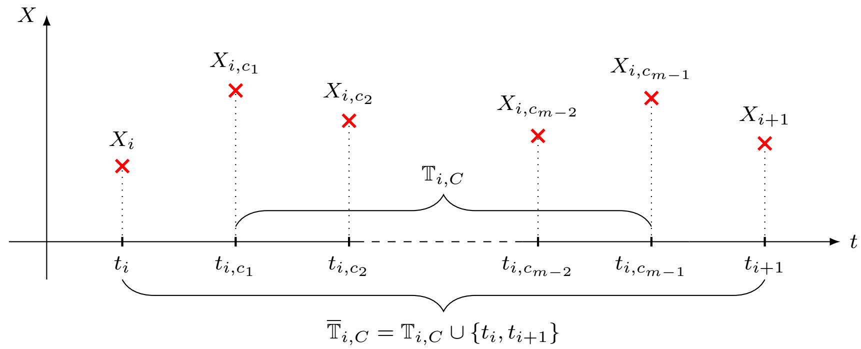
1) *a fixed time scale $\mathbb{T}_\eta$ of the form (III.2.1) called a control time scale which induces time scales $\mathbb{T}_{i,C}$ over $I_i^\pm$ defined by*

$$\mathbb{T}_{i,C} = \{t_{i,c_j} = t_i + \eta_j h,\ \eta_j \in \mathbb{T}_\eta \setminus \{0, 1\},\ j = 1, \ldots, m-1\},$$

where $i = 0, \ldots, N-1$ and we denote

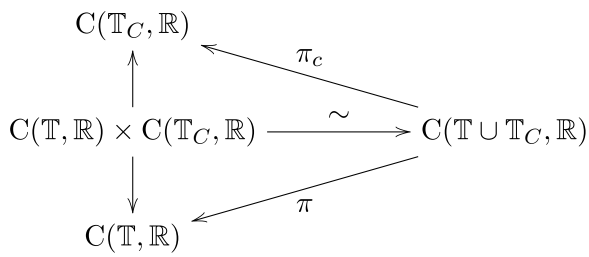
$$\mathbb{T}_C = \bigcup_{i=0}^{N-1} \mathbb{T}_{i,C}.$$

2) *a function $X_c \in \mathrm{C}(\mathbb{T}_C, \mathbb{R})$ called a control function of $\mathbb{T}_C$.*



Definition III.3. *Let $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ and let $(\mathbb{T}_\eta, X_c)$ be a control of order m over $\mathbb{T}$. The extension of X with respect to $(\mathbb{T}_\eta, X_c)$ is the function $Z \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_C, \mathbb{R})$ defined by*

$$Z_{|_{\mathbb{T}}} = X, \quad \textit{and} \quad Z_{|_{\mathbb{T}_C}} = X_c.$$



In what follows, we sometimes write $Z \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_C, \mathbb{R})$ or equivalently $(X, X_c) \in \mathrm{C}(\mathbb{T}, \mathbb{R}) \times \mathrm{C}(\mathbb{T}_C, \mathbb{R})$ by using isomorphism $\sim$.

Definition III.4. *We denote by $\ell_{j,i}$ the function defined for all $t \in I_i$ by*

$$\ell_{j,i}(t) = \ell_j\Big(\frac{t - t_i}{h}\Big), \quad i = 1, \ldots, N-1, \quad j = 0, \ldots, m.$$

Definition III.5 (Interpolation map of degree m)**.** *Let $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ and let $(\mathbb{T}_\eta, X_c)$ be a control of order m over $\mathbb{T}$. An interpolation of order m of X with respect to $(\mathbb{T}_\eta, X_c)$ is a map $\kappa_m : \mathrm{C}(\mathbb{T} \cup \mathbb{T}_C, \mathbb{R}) \longrightarrow \mathscr{P}_m([a, b], \mathbb{R})$ defined for all $t \in I_i$ by*

$$\kappa_{m,i}\big(Z\big)\,(t) = \sum_{j=0}^{m} \ell_{j,i}(t) Z(t_{i,c_j}) = \ell_{0,i}(t) X(t_i) + \ell_{m,i}(t) X(t_{i+1}) + \sum_{j=1}^{m-1} \ell_{j,i}(t) X_c(t_{i,c_j}), \qquad \text{(III.3.1)}$$

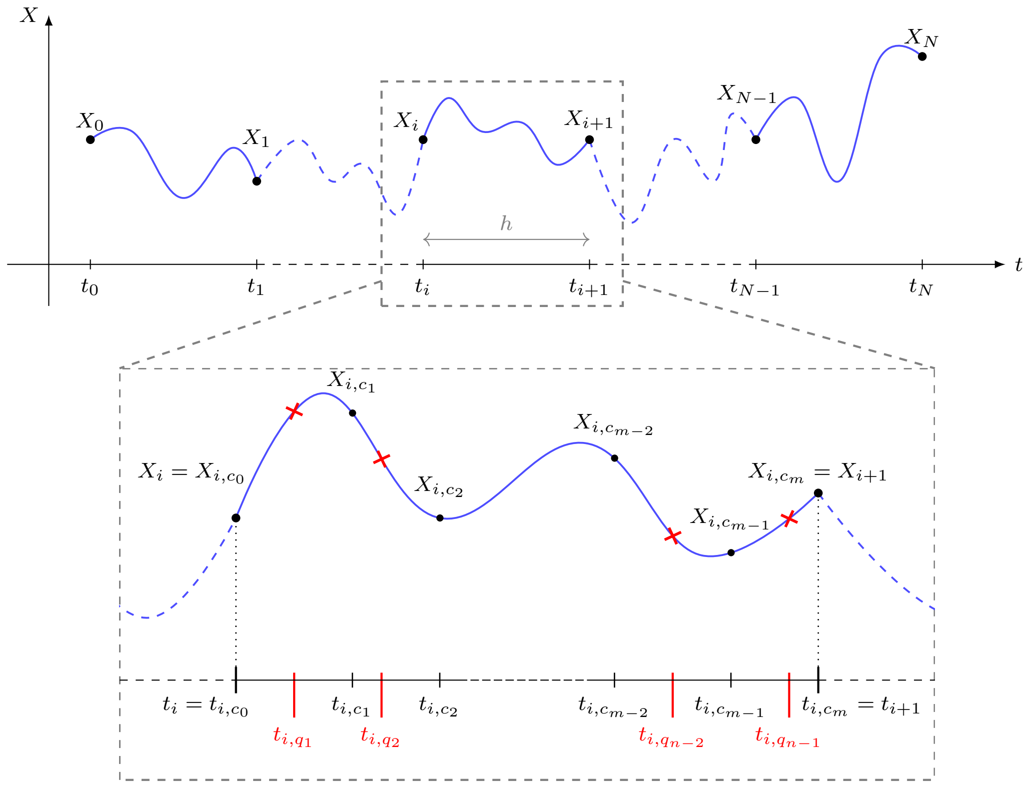
where $t \in I_i$ and Z is the extension of X with respect to $(\mathbb{T}_\eta, X_c)$.

It is possible to define the interpolation of Z not only on I_i but also on $[a, b]$. Indeed, if we take $\kappa_{m,i}\big(Z\big)$ as the interpolation on I_i, then we have for all $t \in [a, b]$

$$\kappa_m\big(Z\big)\,(t) = \sum_{i=1}^{N-1} \kappa_{m,i}\big(Z\big)\,(t)\,\mathbb{1}_{I_i^+}(t).$$

Example III.1. For $m = 1$, we have $\mathbb{T}_C = \emptyset$ and then $\kappa_{m,i}\big(Z\big)\,(t)$ is just the classical linear interpolation, that is

$$\kappa_{1,i}\big(Z\big)\,(t) = \frac{1}{h}\big[(t_{i+1} - t) X(t_i) + (t - t_i) X(t_{i+1})\big] \quad \text{for} \quad t \in I_i, \quad i = 0, \ldots, N-1.$$



III.3.1 Properties of interpolations

Let us write $\kappa_{m,i}\big(Z\big)$ as a sum of two interpolations as follows

$$\begin{aligned}\kappa_{m,i}\big(Z\big)\,(t) &= \ell_{0,i}(t)X(t_i) + \ell_{m,i}(t)X(t_{i+1}) + \sum_{j=1}^{m-1} \ell_{j,i}(t)X_c(t_{i,c_j})\\ &:= \kappa_x\big(X\big)(t) + \kappa_c\big(X_c\big)(t).\end{aligned}$$

The interpolation $\kappa_m\big(Z\big)$ has the following properties

Property III.1. *Let $X \in \mathrm{C}(\mathbb{T},\mathbb{R})$, $X_c \in \mathrm{C}(\mathbb{T}_C,\mathbb{R})$ and let $Z = (X, X_c)$ be the extension of X, we have the following properties*

1. *for all $t \in I_i$, $\kappa_m\big(Z\big)(t) = \kappa_x\big(X\big)(t) + \kappa_c\big(X_c\big)(t)$*

2. *for all $t \in I_i$ and for all $\alpha, \beta \in \mathbb{R}$*

$$\begin{aligned}\kappa_x\big(\alpha X + \beta Y\big)(t) &= \alpha\kappa_x\big(X\big)(t) + \beta\kappa_x\big(Y\big)(t), \quad \forall X, Y \in \mathrm{C}(\mathbb{T},\mathbb{R})\\ \kappa_c\big(\alpha X_c + \beta Y_c\big)(t) &= \alpha\kappa_c\big(X_c\big)(t) + \beta\kappa_c\big(Y_c\big)(t), \quad \forall X_c, Y_c \in \mathrm{C}(\mathbb{T}_C,\mathbb{R})\\ \kappa_m\big(\alpha Z + \beta W\big)(t) &= \alpha\kappa_m\big(Z\big)(t) + \beta\kappa_m\big(W\big)(t), \quad \forall Z, W \in \mathrm{C}(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})\end{aligned}$$

3. *for all $t_i \in \mathbb{T}$, $\kappa_m\big(X\big)(t_i) = X(t_i)$ and $\kappa_m\big(X\big)(t_{i+1}) = X(t_{i+1})$.*

III.4 High-order discrete derivative

III.4.1 High-order derivative

Definition III.6 (Discrete derivatives). *Let $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$, $(\mathbb{T}_\eta, X_c)$ be a control of order m and Z the extension of X with respect to X_c, the "discrete" derivative of Z is defined by*

$$\Delta_{m,i}\big(Z\big)(t) := \frac{d^+}{dt}\left(\kappa_{m,i}\big(Z\big)\right)(t) = \sum_{j=0}^{m} \ell'_{j,i}(t) Z(t_{i,c_j}), \quad \forall t \in I_i^+.$$

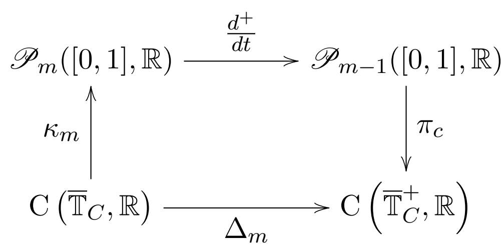
We have for all $t \in [a, b]$

$$\Delta_m\big(Z\big)(t) = \sum_{i=1}^{N-1} \Delta_{m,i}\big(Z\big)(t) \mathbb{1}_{I_i^+}(t).$$

Example III.2. For $m = 1$, we have $\mathbb{T}_C = \emptyset$ and then $\Delta_{1,i}$ (or simply) Δ_i is the usual forward discrete operator, namely

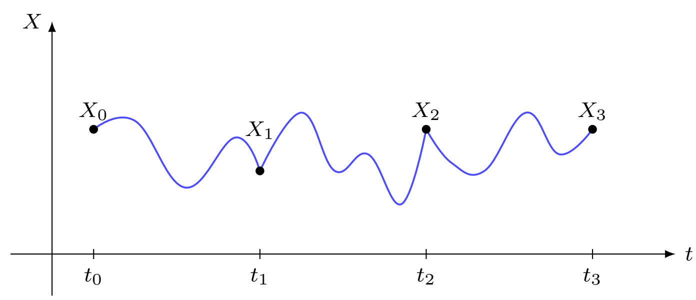
$$\Delta_i\big(X\big)\,(t) := \Delta_+\big[X\big](t_i) = \frac{1}{h}\big[X(t_{i+1}) - X(t_i)\big] \quad \text{for} \quad t \in I_i, \quad i = 0, \ldots, N-1.$$

We have the corresponding commutative diagram



The word "discrete" refer to the function Z which is only defined on a discrete time scale and not on the notation of Δ_m which is a piecewise continuous function.

Remark III.1. *The interpolation mapping κ_m is not differentiable for $t \in \mathbb{T}$. This is a reason for using the right derivative over $\mathbb{T}^+$.*



III.4.2 Properties of derivative operator

Consider the following notation

$$\begin{aligned}\Delta_m\big(Z\big)(t) := \frac{d}{dt}\left(\kappa_m\big(Z\big)\right)(t) &= \frac{d}{dt}\left(\kappa_c\big(X\big)\right)(t) + \frac{d}{dt}\left(\kappa_c\big(X_c\big)\right)(t) \\ &= \Delta_x\big(X\big)(t) + \Delta_c\big(X_c\big)(t).\end{aligned}$$

We have the following properties:

Property III.2. *Let $X \in \mathrm{C}(\mathbb{T},\mathbb{R})$, $X_c \in \mathrm{C}(\mathbb{T}_C,\mathbb{R})$ and let $Z = (X, X_c)$ be the extension of X, we have the following properties*

1. $\Delta_m\big(Z\big)(t) = \Delta_x\big(X\big)(t) + \Delta_c\big(X_c\big)(t)$

2. $\forall \alpha, \beta \in \mathbb{R}$

$$\Delta_x\big(\alpha X + \beta Y\big) = \alpha\Delta_x\big(X\big) + \beta\Delta_x\big(X_c\big), \quad \forall X, Y \in \mathrm{C}(\mathbb{T},\mathbb{R})$$
$$\Delta_c\big(\alpha Y_c + \beta Y_c\big)(t) = \alpha\Delta_c\big(X_c\big) + \beta\Delta_c\big(Y_c\big), \quad \forall X_c, Y_c \in \mathrm{C}(\mathbb{T},\mathbb{R})$$

3. *for all $t_i \in \mathbb{T}$, $\kappa_{m,i}\big(X\big)(t_i) = X(t_i)$ and $\kappa_{m,i}\big(X\big)(t_{i+1}) = X(t_{i+1})$.*

III.5 High-order discrete anti-derivatives

III.5.1 Reminder about the usual $\Delta_\pm$-integrals

Let $\mathbb{T}$ be discrete time scales and $f \in \mathrm{C}(\mathbb{T},\mathbb{R})$, we recall that

$$\int_a^b f(t)\ \Delta_+ t = h\sum_{i=0}^{N-1} f(t_i) \quad \text{and} \quad \int_a^b f(t)\ \Delta_- t = h\sum_{i=1}^{N} f(t_i). \tag{III.5.1}$$

The $\Delta_\pm$-integrals have the following properties (see [5, Theorem 2.8])

Proposition III.1. *Let $f \in \mathrm{C}(\mathbb{T},\mathbb{R})$, then for all $a, b \in \mathbb{T}$ with $a \leq b$ we have have*

$$\begin{aligned}
\int_a^b f(t)\ \Delta_+ t &= hf^\rho(b) + \int_a^{\rho(b)} f(t)\ \Delta_+ t,\\
\int_a^b f(t)\ \Delta_+ t &= hf(a) + \int_{\sigma(a)}^{b} f(t)\ \Delta_+ t,\\
\int_a^b f(t)\ \Delta_- t &= hf(b) + \int_a^{\rho(b)} f(t)\ \Delta_- t,\\
\int_a^b f(t)\ \Delta_- t &= hf^\sigma(a) + \int_{\sigma(a)}^{b} f(t)\ \Delta_- t.
\end{aligned} \tag{III.5.2}$$

We have the relationship between the $\Delta_\pm$-integrals (see [43, Proposition 7])

Proposition III.2. *Let $f \in \mathrm{C}(\mathbb{T},\mathbb{R})$, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\ \Delta_+ t = \int_a^b f^\rho(t)\ \Delta_- t \quad \textit{and} \quad \int_a^b f(t)\ \Delta_- t = \int_a^b f^\sigma(t)\ \Delta_+ t. \tag{III.5.3}$$

Using the definition of $\Delta_\pm$-integrals (equations (III.5.1)) and the two previous propositions, one can obtain the following result:

Proposition III.3. *Let $f, g \in \mathrm{C}(\mathbb{T},\mathbb{R})$, then for all $a, b \in \mathbb{T}$ with $a \leq b$ we have*

$$\int_{\sigma(a)}^b f(t)\ \Delta_+ t + \int_a^{\rho(b)} g(t)\ \Delta_- t = \int_{\sigma(a)}^b \big(f(t) + g(t)\big)\ \Delta_+ t = \int_a^{\rho(b)} \big(f(t) + g(t)\big)\ \Delta_- t \tag{III.5.4}$$

III.5.2 High-order discrete anti-derivatives

We want to define a discrete operator extending the classical Δ anti-derivative on time scales. A family of anti-derivatives is defined using the classical notion of quadrature that we adapt to our framework.

Definition III.7. *A quadrature $(\mathbb{T}_q, w)$ of order n over $\mathbb{T}$ is the data of*

1. *A time scale called quadrature $\mathbb{T}_q$ over $[0,1]$ defined by*

$$\mathbb{T}_q = \{q_i,\ i = 0, \dots, n\}, \quad \text{with} \quad q_0 = 0,\ q_n = 1.$$

2. *A weight function $w \in \mathrm{C}(\mathbb{T}_q, [0,1])$ such that*

$$\sum_{i=0}^{n} w(q_i) := \sum_{i=0}^{n} w_i = 1$$

Definition III.8. *Let $(\mathbb{T}_q, w)$ be a quadrature of order n over $\mathbb{T}$. We denote by*

$$\mathbb{T}_{i,Q} = \{t_{i,q} = t_i + q_j h,\ q_j \in \mathbb{T}_q \setminus \{0,1\},\ j = 1, \dots, n-1\},$$

where $i = 0, \dots, N-1$ and we denote

$$\mathbb{T}_Q = \bigcup_{i=0}^{N-1} \mathbb{T}_{i,Q}.$$

Definition III.9. *Let $t \in \mathbb{T} \cup \mathbb{T}_Q$, we define the forward jump operator $\sigma^n_{\mathbb{T}}$ over $\mathbb{T} \cup \mathbb{T}_Q$ by*

$$\sigma^n_{\mathbb{T}}(t) = \begin{cases} \sigma(t_i) = t_{i+1} & \text{if} \quad t = t_i \in \mathbb{T} \\ \sigma^n(t_{i,q_j}) = t_{i,q_{j+1}} & \text{if} \quad t = t_{i,q_j} \in \mathbb{T}_Q, \end{cases}$$

Similarly, we define the backward jump operator $\rho^n_{\mathbb{T}}$ over $\mathbb{T} \cup \mathbb{T}_Q$ by

$$\rho^n_{\mathbb{T}}(t) = \begin{cases} \rho(t_i) = t_{i-1} & \text{if} \quad t = t_i \in \mathbb{T} \\ \rho^n(t_{i,q_j}) = t_{i,q_{j-1}} & \text{if} \quad t = t_{i,q_j} \in \mathbb{T}_Q, \end{cases}$$

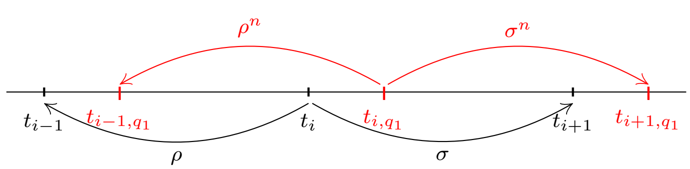


Figure III.1: The jump operators $\sigma^n_{\mathbb{T}}$ and $\rho^n_{\mathbb{T}}$.

Definition III.10. *Let $f \in \mathrm{C}(\mathbb{T}_Q, \mathbb{R})$, we define on for $t \in \mathbb{T}_{i,Q}$*

$$\int_{t_{i,q_j}}^{t_{i,q_{j+1}}} f(t)\ \Delta_{q,+} t = h f(t_{i,q_j}) \quad \text{and} \quad \int_{t_{i,q_j}}^{t_{i,q_{j+1}}} f(t)\ \Delta_{q,-} t = h f(t_{i,q_{j+1}}),$$

for all $i = 0, \dots, N-1$.

III.5.3 Quadrature formula

Definition III.11. *Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$. The $(\mathbb{T}_q, w)$-integral of f over $\mathbb{T}$ denoted by $\int_a^t f(s)\ \Delta_{q,w} s$ is defined for all $t = t_k$ by*

$$\int_a^{t_k} f(t)\ \Delta_{q,w} t = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(t)\ \Delta_{q,w} t, \tag{III.5.5}$$

where

$$\int_{t_i}^{t_{i+1}} f(t)\ \Delta_{q,w} t = h \sum_{j=0}^{n} w_j f(t_i + q_j h) := \int_{t_i}^{t_{i+1}} w(t) f(t) \Delta_q t. \tag{III.5.6}$$

III.5.4 Properties of antiderivative operator

Property III.3. *Let $f, g \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $\lambda \in \mathbb{R}$, we have the following properties*

$$\int (\lambda f + g)\ \Delta_{q,w} t = \lambda \int f\ \Delta_{q,w} s + \int g\ \Delta_{q,w} t \tag{III.5.7}$$

With the previous definition, one can rewrite the quadrature formula (III.5.6) as follows

Property III.4. *Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$, we have*

$$\begin{aligned}
\int_{t_i}^{\sigma(t_i)} f(t)\ \Delta_{q,w} t &= w_0 \int_{t_i}^{\sigma(t_i)} f(t)\ \Delta_+ s + \sum_{j=1}^{n-1} w_j \int_{t_{i,q_j}}^{t_{i,q_{j+1}}} f(t)\ \Delta_{q,+} t + w_n \int_{t_i}^{\sigma(t_i)} f(t)\ \Delta_- t \\
&= \sum_{j=0}^{n-1} w_j \int_{t_{i,q_j}}^{t_{i,q_{j+1}}} f(s)\ \Delta_{q,+} t + w_n \int_{t}^{\sigma(t)} f(s)\ \Delta_- t
\end{aligned}$$

for all $i = 0, \ldots, N-1$ and such that

$$\int_a^b f(t)\ \Delta_{q,w} t = \int_a^b w(t) f(t) \Delta_+ t + \int_a^b w(t) f(t) \Delta_- t + \int_{\mathbb{T}_Q} w(t) f(t) \Delta_{q,+} t, \tag{III.5.8}$$

for all $t \in \mathbb{T} \cup \mathbb{T}_Q$.

We consider the following commutative diagram to illustrate this definition

$$\begin{array}{ccc}
\mathscr{P}_m([a,b],\mathbb{R}) & \xrightarrow{\int_a^t} & \mathscr{P}_{m+1}([a,b],\mathbb{R}) \\
\uparrow \kappa_m & & \downarrow \pi_q \\
\mathrm{C}\left(\overline{\mathbb{T}}_C, \mathbb{R}\right) & \xrightarrow[(\mathbb{T}_q, w)\text{-integral}]{} & \mathrm{C}\left(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R}\right)
\end{array}$$

By introducing the transformation $s = hx + t_i$, one can transform the interval $[t_i, t_{i+1}]$ to $[0, 1]$, so we have

$$f(s) = f(hx + t_i) = g(x)$$

and the formula (III.5.6) becomes

$$\int_0^1 g(x)\ \Delta_{q,w}x := h\sum_{j=0}^{n} w_j g(q_j h). \tag{III.5.9}$$

A notion of exactness for $(\mathbb{T}_q, w)$-integral is defined as follows (see [86, Definition 3.2.1, p.170]).

Definition III.12. *A $(\mathbb{T}_q, w)$-integral is said to be exact on $\mathscr{P}_m$ if*

$$\int_a^b f(t)dt - \int_a^b f(t)\Delta_{q,w}t = 0, \quad \forall f \in \mathscr{P}_m. \tag{III.5.10}$$

Let us denote by $\pi_c : \mathscr{P}_m([0,1],\mathbb{R}) \longrightarrow \mathrm{C}\left(\overline{\mathbb{T}}_C, \mathbb{R}\right)$ the map defined by the restriction of a polynomial $P \in \mathscr{P}_m$ to $\overline{\mathbb{T}}_C$, we have the following properties.

Property III.5. *Let $P \in \mathscr{P}_m([0,1],\mathbb{R})$. Then we have,*

$$\kappa_m \circ \pi_c \circ P = P. \tag{III.5.11}$$

We then deduce the following exactness result:

Property III.6. *Let $f \in \mathrm{C}\left(\overline{\mathbb{T}}_C, \mathbb{R}\right)$ and $P \in \mathscr{P}_m([0,1],\mathbb{R})$. If $f = \pi_c \circ P$, then*

$$\int_0^1 \pi_q \circ P(t)\Delta_{q,w}t := h\sum_{k=0}^{n} w_k P(q_k) = \int_0^1 P(t)dt.$$

III.5.4.1 High-order discrete Dubois-Raymond Lemma

Theorem III.1 (High-order Discrete Dubois-Raymond). *Let $f \in \mathrm{C}(\mathbb{T}\cup\mathbb{T}_Q, \mathbb{R})$ such that*

$$\int_a^b f(t)v(t)\ \Delta_+ s = 0$$

for all $v \in \mathrm{C}_0(\mathbb{T},\mathbb{R})$. Then,

$$\sum_{k=0}^{n} w_k f(t_{i,q_k}) = 0 \quad \textit{for} \quad i = 1,\dots,N-1.$$

III.6 Properties of high-order discrete derivatives and antiderivatives

III.6.1 Integration by parts formula

We have the definition of the functions $f^{\circ,+}$ and $f^{\circ,-}$.

Definition III.13. *Let $f \in \mathrm{C}(\mathbb{T}\cup\mathbb{T}_Q, \mathbb{R})$, we define the functions $f^{\circ,-}$ and $f^{\circ,+}$ for all $t \in \mathbb{T}\cup\mathbb{T}_Q$ as*

$$f^{\circ,-}(t) = \ell_0(t)f(t) + \ell_m(t)f^{\rho^n_{\mathbb{T}}}(t) \quad \textit{and} \quad f^{\circ,+}(t) = \ell_0(t)f(t) + \ell_m(t)f^{\sigma^n_{\mathbb{T}}}(t),$$

then

$$\Delta_q f^{\circ}(t) = \ell_0'(t)f(t) + \ell_m'(t)f^{\rho^n_{\mathbb{T}}}(t).$$

Now, for all $t \in \mathbb{T} \cup \mathbb{T}_{i,Q}$, we have

$$f^{\circ,-}(t_{i,q_j}) = \ell_0(q_j) f\left(t_{i,q_j}\right) + \ell_m(q_j) f\left(t_{i-1,q_j}\right), \quad q_j \in \mathbb{T}_q.$$

Theorem III.2. *Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ we have*

$$\int_a^b f(t)\kappa_x\big(X\big)(t)\Delta_{q,w}t = \int_{\sigma(a)}^b X(t)\left[\int_t^{\sigma(t)} f^{\circ,-}(s)\Delta_{q,w}s\right]\Delta_+ t + \mathbb{F}_0, \tag{III.6.1}$$

$$\int_a^b f(t)\Delta_x X(t)\Delta_{q,w}t = \int_{\sigma(a)}^b X(t)\left[\int_t^{\sigma(t)} \Delta_q f(s)\Delta_{q,w}s\right]\Delta_+ t + \mathbb{G}_0, \tag{III.6.2}$$

where

$$\mathbb{F}_0 = X(t_0)\int_{t_0}^{t_1} \ell_0(t) f(t)\Delta_{q,w}t + X(t_N)\int_{t_{N-1}}^{t_N} \ell_m(t) f(t)\Delta_{q,w}t$$

and

$$\mathbb{G}_0 = X(t_0)\int_{t_0}^{t_1} \ell_0'(t) f(t)\Delta_{q,w}t + X(t_N)\int_{t_{N-1}}^{t_N} \ell_m'(t) f(t)\Delta_{q,w}t.$$

The proof is based on the following observation.

Lemma III.1. *Let $f \in \mathrm{C}(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $X \in \mathrm{C}(\mathbb{T}, \mathbb{R})$. We define for all $t \in \mathbb{T}^+$ the following function*

$$t \longmapsto F_0(t) = \int_t^{\sigma(t)} f(s)\ell_0(s)\Delta_{q,w}s$$

$$t \longmapsto F_m(t) = \int_t^{\sigma(t)} f(s)\ell_m(s)\Delta_{q,w}s$$

Then,

$$\int_a^b \big(X(t)F_0(t) + X^\sigma(t)F_m(t)\big)\Delta_+ t = hX(a)F_0(a) + hX(b)F_m^\rho(b) + \int_{\sigma(a)}^b X(t)\big(F_0(t) + F_m^\rho(t)\big)\Delta_+ t.$$

Proof. From Proposition (III.1), the second equation of (III.5.2) gives

$$I_1 = \int_a^b X(t)F_0(t)\Delta_+ t = hX(a)F_0(a) + \int_{\sigma(a)}^b X(t)F_0(t)\Delta_+ t.$$

Using the first equation of (III.5.2) for the left equation of (III.5.3) gives

$$\begin{aligned} I_2 = \int_a^b X^\sigma(t)F_m(t)\Delta_+ t &= hX(b)F_m^\rho(b) + \int_a^{\rho(b)} X^\sigma(t)F_m(t)\Delta_+ t \\ &= hX(b)F_m^\rho(b) + \int_a^{\rho(b)} X(t)F_m^\rho(t)\Delta_- t. \end{aligned}$$

We obtain the result by forming the sum $I_1 + I_2$ and using Proposition III.3. □

Now, we make the proof for equation (III.6.1) in the previous theorem, a similar computation can be done for equation (III.6.2).

Proof of Theorem III.2. Let $f \in C(\mathbb{T} \cup \mathbb{T}_Q, \mathbb{R})$ and $X \in C(\mathbb{T}, \mathbb{R})$. We have

$$\int_{t_i}^{t_{i+1}} f(t)\kappa_{x,i}(X)(t)\Delta_{q,w}t = hX(t_i)\int_{t_i}^{t_{i+1}} f(t)\ell_{0,i}(t)\Delta_{q,w}t + hX^\sigma(t_i)\int_{t_i}^{t_{i+1}} f(t)\ell_{m,i}(t)\Delta_{q,w}t$$
$$= hX(t_i)F_0(t_i) + hX^\sigma(t_i)F_m(t_i).$$

By summation in i from 0 to $N-1$, one obtains

$$\int_a^b f(t)\kappa_x(X)(t)\Delta_{q,w}t = \int_a^b \left(X(t)F_0(t) + X^\sigma(t)F_m(t)\right)\Delta_+ t.$$

The proof is completed by using directly previous Lemma and the following formula

$$F_m^\rho(t) = \int_{\rho(t)}^t f(s)\ell_m(s)\Delta_{q,w}s = \int_t^{\sigma(t)} f^{\rho_{\mathbb{T}}^n}(s)\ell_m(s)\Delta_{q,w}s.$$

□

III.6.2 Fundamental theorem of high-order time scale calculus

Constructing discrete operators approximating the classical derivative and the classical antiderivative ask for the preservation at the discrete level of the fundamental theorem of differential calculus. The next results show that we preserve the important property of the two operators at the discrete level.

Theorem III.3. *Let $f \in C(\overline{\mathbb{T}}_C, \mathbb{R})$ and $(\mathbb{T}_q, w)$ be a quadrature of order n over $[0,1]$. Then,*

$$\int_0^1 \kappa_m \circ \Delta_m\big(f\big)(t)\ \Delta_{q,w}t = f(1) - f(0).$$

Proof. We have $\Delta_m\big(f\big) = \pi_c \circ \frac{d}{dt} \circ \kappa_m\big(f\big)$ and $\frac{d}{dt} \circ \kappa_m\big(f\big) \in \mathscr{P}_{m-1}([0,1], \mathbb{R})$. Then, using Property V.2, one obtain

$$\kappa_m \circ \Delta_m\big(f\big) = \frac{d}{dt} \circ \kappa_m\big(f\big).$$

As $f = \pi_c \circ P$, we deduce by using Property III.6 that

$$\int_0^1 \pi_q \circ \frac{d}{dt}\left(\kappa_m\big(f\big)\right)(t)\ \Delta_{q,w}t = \int_0^1 \frac{d}{dt}\left(\kappa_m\big(f\big)\right)(t)\ dt = \kappa_m\big(f\big)(1) - \kappa_m\big(f\big)(0) = f(1) - f(0).$$

□

Theorem III.4. *Let $f \in C(\overline{\mathbb{T}}_C, \mathbb{R})$ and $(\mathbb{T}_q, w)$ be a quadrature of order n over $[0,1]$. Then,*

$$\Delta_m \circ \int_0^t \kappa_m\big(f\big)(s)\ \Delta_{q,w}s = f.$$

Proof. Let f satisfies the conditions of Property III.6. As $\int_0^t \kappa_m\big(f\big)(s)ds \in \mathscr{P}_{m+1}([0,1], \mathbb{R})$, Then

$$\kappa_m \circ \int_0^t \kappa_m\big(f\big)(s)ds = \int_0^t \kappa_m\big(f\big)(s)ds$$

Using the definition of Δ_m, we deduce

$$\pi_c \circ \frac{d}{dt} \circ \kappa_m \circ \int_0^t \kappa_m\big(f\big)(s)ds = \pi_c \circ \frac{d}{dt} \circ \int_0^t \kappa_m\big(f\big)(s)ds = \pi_c \circ \kappa_m\big(f\big)(t) = f$$

□

III.7 High order time scale calculus of variations

Definition III.14. *The discrete Lagrangian functional of order (m,n) over $\mathbb{T}$ with respect to a $(\mathbb{T}_\eta, C)$ control of order m and a $(\mathbb{T}_q, w)$ quadrature of order n is defined for all $X \in \mathrm{C}(\mathbb{T},\mathbb{R})$ and $X_c \in \mathrm{C}(\mathbb{T}_C,\mathbb{R})$ by*

$$\mathscr{L}_{\eta,q,w}(X,X_c) = \int_a^b L\left(\kappa_m\big(Z\big)(t), \Delta_m\big(Z\big)(t)\right)\ \Delta_{q,w}t, \tag{III.7.1}$$

where Z is the extension of X with respect to $(\mathbb{T}_\eta, C)$.

In the following, we will write $\mathscr{L}_w$ instead of $\mathscr{L}_{\eta,q,w}$.

III.7.1 Discrete calculus of variations

Let $\mathrm{C}_0\left(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}\right)$ be a subset of $\mathrm{C}\left(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}\right)$ defined by

$$\mathrm{C}_0\left(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}\right) = \left\{G \in \mathrm{C}\left(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R}\right),\ G_{|_\mathbb{T}}(a) = G_{|_\mathbb{T}}(b) = 0\right\}.$$

Definition III.15. *The Fréchet derivative of $\mathscr{L}_w$ at Z along a direction $H \in \mathrm{C}_0(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$ is defined by*

$$D\mathscr{L}_w(Z)(H) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\mathscr{L}_w(Z+\epsilon H) - \mathscr{L}_w(Z)\big).$$

Definition III.16. *A function $Z \in \mathrm{C}(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$ is said to be a critical point of $\mathscr{L}_w$ if $D\mathscr{L}_w(Z)(H) = 0$ for all variation $H \in \mathrm{C}_0(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$.*

We first compute the Fréchet derivative of $\mathscr{L}_w$.

Lemma III.2. *Let $\star(t) = \big(\kappa_m\big(Z\big)(t), \Delta_m\big(Z\big)(t)\big)$ and let $H = (V,V_c) \in \mathrm{C}_0(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$ be the variation of the extension Z. Then the Fréchet derivative of the functional $\mathscr{L}_w$ is given by*

$$\begin{aligned} D\mathscr{L}_w(Z)(H) &= \int_a^b \left[\frac{\partial L}{\partial x}\left(\star(t)\right)\kappa_m\big(H\big)(t) + \frac{\partial L}{\partial v}\left(\star(t)\right)\Delta_m\big(H\big)(t)\right]\ \Delta_{q,w}t \\ &= \int_a^b \left[\frac{\partial L}{\partial x}\left(\star(t)\right)\kappa_x\big(V\big)(t) + \frac{\partial L}{\partial v}\left(\star(t)\right)\Delta_x(V)(t)\right. \\ &\qquad\qquad \left. + \frac{\partial L}{\partial x}\left(\star(t)\right)\kappa_c\big(V_c\big)(t) + \frac{\partial L}{\partial v}\left(\star(t)\right)\Delta_c(V_c)(t)\right]\Delta_{q,w}t. \end{aligned}$$

Proof. Let $\epsilon > 0$ be a small parameter and let $H \in \mathrm{C}_0(\mathbb{T}\cup\mathbb{T}_C,\mathbb{R})$. We have

$$\mathscr{L}_w(Z+\epsilon H) = \int_a^b L\left(\kappa_m\big(Z+\epsilon H\big)(t), \Delta_m\big(Z+\epsilon H\big)(t)\right)\ \Delta_{q,w}t.$$

The linearity of k_m and Δ_m as in Properties III.1 and III.2 lead to

$$\mathscr{L}_w(Z+\epsilon H) = \int_a^b L\left(\kappa_m\big(Z\big)(t) + \epsilon\kappa_m\big(H\big)(t), \Delta_m\big(Z\big)(t) + \epsilon\Delta_m\big(H\big)(t)\right)\ \Delta_{q,w}t.$$

With the help of a Taylor expansion, we obtain the Fréchet derivative of the functional $\mathscr{L}_w$

$$D\mathscr{L}_w(Z)(H) = \int_a^b \left[\frac{\partial L}{\partial x}\left(\star(t)\right)\kappa_m\big(H\big)(t) + \frac{\partial L}{\partial v}\left(\star(t)\right)\Delta_m\big(H\big)(t)\right]\ \Delta_{q,w}t.$$

Using again Properties III.1 and III.2, the last expression can be rewritten as

$$D\mathscr{L}_w(Z)(V,V_c) = \int_a^b \left[\frac{\partial L}{\partial x}(\star(t))\,\kappa_x\big(V\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_x(V)(t) \right.$$
$$\left. + \frac{\partial L}{\partial x}(\star(t))\,\kappa_c\big(V_c\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_c(V_c)(t) \right] \Delta_{q,w} t.$$

This completes the proof. □

Now, with the choice $V = 0$ or $V_c = 0$, we have the following results

Proposition III.4. *The critical points of the functional $\mathscr{L}_w$ are solutions of the following integral equations:*

$$\int_a^b \left[\frac{\partial L}{\partial x}(\star(t))\,\kappa_x\big(V\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_x(V)(t) \right] \Delta_{q,w} t = 0, \quad \forall V \in \mathrm{C}_0(\mathbb{T},\mathbb{R}), \tag{III.7.2}$$

$$\int_a^b \left[\frac{\partial L}{\partial x}(\star(t))\,\kappa_c\big(V_c\big)(t) + \frac{\partial L}{\partial v}(\star(t))\,\Delta_c(V_c)(t) \right] \Delta_{q,w} t = 0, \quad \forall V_c \in \mathrm{C}(\mathbb{T}_C,\mathbb{R}). \tag{III.7.3}$$

High-order discrete calculus of variations then induces two different discrete integral necessary and sufficient conditions for a critical point, one governing the structure of the solution X on $\mathbb{T}$ and the other one related to the control function.

The last step toward a high-order discrete Euler-Lagrange equation is obtained using the high-order discrete integration by part formula.

Theorem III.5. *The critical points of the functional* (III.7.1) *correspond to the solutions of the following Euler-Lagrange equations for all $t \in \mathbb{T}^{\pm}$*

$$\int_t^{\sigma(t)} \left[\left(\frac{\partial L}{\partial x}(\star(s)) \right)^{\circ,-} + \Delta_q \left(\frac{\partial L}{\partial v}(\star(s)) \right)^{\circ,-} \right] \Delta_{q,w} s = 0 \tag{III.7.4}$$

$$\int_t^{\sigma(t)} \ell_j(s) \frac{\partial L}{\partial x}(\star(s))\,\Delta_{q,w} s = -\frac{1}{h} \int_t^{\sigma(t)} \ell_j'(s) \frac{\partial L}{\partial v}(\star(s)\,\Delta_{q,w} s, \quad 1 \leq j \leq m-1. \tag{III.7.5}$$

Proof. Starting with equation (III.7.4). The integration by parts using Theorem III.2 gives

$$\int_{\sigma(a)}^b V(t) \left(\int_t^{\sigma(t)} \left[\left(\frac{\partial L}{\partial x}(\star(s)) \right)^{\circ} + \Delta_q \left(\frac{\partial L}{\partial v}(\star(s)) \right)^{\circ} \right] \Delta_{q,w} s \right) \Delta_+ t + \mathbb{F}_0 + \mathbb{G}_0 = 0.$$

As $V \in \mathrm{C}_0(\mathbb{T},\mathbb{R})$, we obtain

$$\int_{\sigma(a)}^b V(t) \left(\int_t^{\sigma(t)} \left[\left(\frac{\partial L}{\partial x}(\star(s)) \right)^{\circ} + \Delta_q \left(\frac{\partial L}{\partial v}(\star(s)) \right)^{\circ} \right] \Delta_{q,w} s \right) \Delta_+ t = 0.$$

Therefore, we conclude the result by using Dubois-Reymond lemma. □

III.7.2 Example

Let us take the following special cases of the functional $\mathscr{L}_w$.

1. The functional $\mathscr{L}_w$ without control points with one quadrature point $q = 0$ reduces to the classical discrete functional (see Definition II.14, Chapter II)

$$\mathscr{L}(X) = \int_a^b L\left(X(t), \Delta_+ X(t)\right) \; \Delta t = h \sum_{i=0}^{N-1} L\left(X(t_i), \Delta_+ X(t_i)\right)$$

2. The functional $\mathscr{L}_w$ without control points with one quadrature point $q = 1$ reduces to the shifted discrete functional (see Equation (V.2.13), Chapter V)

$$\mathscr{L}(X) = \int_a^b L\left(X(t), \Delta_+ X(t)\right) \; \Delta t = h \sum_{i=0}^{N-1} L\left(X(t_{i+1}), \Delta_+ X(t_i)\right)$$

3. The functional $\mathscr{L}_w$ without control points with one quadrature point $q = \frac{1}{2}$ reduces to the midpoint rule

$$\mathscr{L}(X) = h \sum_{i=0}^{N-1} L\left(\frac{X(t_i) + X^{\sigma}(t_i)}{2}, \Delta_+ X(t_i)\right)$$

Part B

Time-scale Noether's Theorems for Lagrangian and Hamiltonian Systems

Chapter IV

Reminder about Lagrangian and Hamiltonian Noether's Theorems

In this chapter, we present briefly the classical Noether's theorem for both Lagrangian and Hamiltonian systems, For a deeper discussion of Noether's theorem we refer the reader to ([9], [17], [58], [83]).

IV.1 Introduction

Conservation laws (or first integrals, constants of motion, conserved quantities, etc.) are important principles for both mathematics as well as theoretical physics. It is commonly acknowledged that conservation laws play a central role in dynamical systems since they describe conserved quantities such as energy momentum, angular momentum, etc. More than that, they can be used to simplify ordinary differential equations to be solved by quadrature and to investigate stability and complete integrability of systems.

Of course, finding conservation laws is not so easy for a general system of differential equations. However there exists a class of systems for which a strategy can be found, namely, systems coming from the variational principle. The two most commonly popular problems of variational principle are Lagrangian and Hamiltonian systems which we will focus on in this chapter.

In 1918, Emmy Noether gave a role between symmetries of Lagrangian system and the existence of conservation laws of the Euler-Lagrange equations (see [58, p.26], Bruce [17, p.208] and Olver [83, p.272]).

The Noether's theorem for Hamiltonian systems, even if it was already contains in the E. Noether's original formulation (see [61, §.5.5]) is not so common as the one for Lagrangian equations. The main difference lies in the fact that not all the transformation groups can be considered but only canonical transformation groups which preserve the Hamiltonian character of the equations under transformations. We refer to the work of A. Mouchet [78] for a very interesting discussion of this theorem.

Organization of the chapter. Section IV.2 and IV.3 is devoted to reminders on a one-parameter group of transformation and the classical Lagrangian Noether's theorem. In Section IV.4, we review briefly some classical results about Hamiltonian systems. The notion of canonical group of transformations was presented in IV.5. In Section IV.6, we recall the Hamiltonian version of the Noether's theorem in the classical case.

IV.2 A one-parameter group of transformations

Let $s \in \mathbb{R}$ and let g_s be a transformation depends smoothly on a real parameter s defined by

$$\begin{array}{rcll} g_s : & [a,b]\times\mathbb{R}^d & \longrightarrow & \mathbb{R}\times\mathbb{R}^d \\ & (t,x) & \longmapsto & g_s(t,x) = \big(g_s^0(t,x), g_s^1(t,x)\big) . \end{array}$$

Definition IV.1. *A family of transformations $\{g_s\}_{s\in\mathbb{R}}$ is said to be one-parameter group of transformations (or of diffeomorphisms) if*

- $g_0 = \mathrm{Id}$ *the identity transformation,*
- *for all $s \in \mathbb{R}$, then $g_s^{-1} = g_{-s}$*
- *for all $g_s, g_\delta \in \{g_s\}_{s\in\mathbb{R}}$, then $g_s \circ g_\epsilon = g_{s+\delta}$.*

The associated *infinitesimal (or local) group action* (see [83, p.51] and [57, p.25]) or transformations is obtained by making a Taylor expansion of g_s around $s = 0$:

$$\begin{aligned} g_s(t,x) &= g_0(t,x) + s\, \frac{\partial g_s(t,x)}{\partial s}\bigg|_{s=0} + O\left(s^2\right) . \\ &= (t,x) + s\left(\frac{\partial g_s^0(t,x)}{\partial s}\bigg|_{s=0}, \frac{\partial g_s^1(t,x)}{\partial s}\bigg|_{s=0} \right) + O\left(s^2\right) \\ &= (t,x) + s\big(\xi(t,x), \zeta(t,x)\big) + O\left(s^2\right) . \end{aligned} \tag{IV.2.1}$$

The infinitesimal generator of the group a differential operator defined by group (see [83, p.27]), that is

$$\mathbf{X} = \xi(t,x)\frac{\partial}{\partial t} + \zeta(t,x)\cdot\frac{\partial}{\partial x} . \tag{IV.2.2}$$

Example IV.1. Let $d = 2$. The translations group (resp. rotations group) are denoted by

$$(t,x,y) \longmapsto (t+s, x_1, x_2) \quad \Big(\text{resp.} \quad (t,x,y) \longmapsto \Big(t, R_s(x,y)^{\mathsf{T}}\Big)\Big) ,$$

and the associated infinitesimals are given by

$$\frac{\partial g_s}{\partial s}\bigg|_{s=0}(t,x_1,x_2) = (1,0,0) \quad \left(\text{resp.} \frac{\partial g_s}{\partial s}\bigg|_{s=0}(t,x_1,x_2) = (0,x_2,-x_1)\right) ,$$

where $(\cdot)^{\mathsf{T}}$ denotes the transpose operator and R_s is the orthogonal the 2×2 matrix defined by

$$R_s = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} .$$

IV.3 Lagrangian Noether's theorem

We define the invariance condition of the functional $\mathcal{L}$.

Definition IV.2 (Variational symmetry). *A Lagrangian functional $\mathcal{L}$ is said to be invariant under the transformation g_s (or g_s is a variational symmetry of $\mathcal{L}$) if for any subinterval $[t_a, t_b] \subset [a, b]$*

$$\int_{t_a}^{t_b} L\big(t, x(t), \dot{x}(t)\big)dt = \int_{\bar{t}_a}^{\bar{t}_b} L\big(\bar{t}, \bar{x}(\bar{t}), \dot{\bar{x}}(\bar{t})\big)d\bar{t}$$

It is worth adding that any symmetry of $\mathcal{L}$ is also a symmetry of the Euler–Lagrange equation. Thus, knowledge of Euler–Lagrange equation symmetries allowing to find all variational symmetries of $\mathcal{L}$ via symmetry criterion using the infinitesimal generator (see [83, p.253]).

Definition IV.3 (Constant of motion). *A function $I(t, x(t))$ is said to be a constant of motion (or first integral, conserved quantity) if and only if*

$$\frac{d}{dt} I\big(\cdot, x(\cdot)\big)(t) = 0 \quad or \quad I(t, x(t)) = constant$$

over all the solutions of the Euler–Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\cdot, x, \dot{x})\right](t) = \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)).$$

Theorem IV.1 (Noether's theorem - Lagrangian systems). *If g_s is a variational symmetry of $\mathcal{L}$, then the function*

$$I(t, x(t)) = \xi(t, x)\left(L(t, x, \dot{x}) - \dot{x} \cdot \frac{\partial L}{\partial v}(t, x, \dot{x})\right) + \zeta(t, x) \cdot \frac{\partial L}{\partial v}(t, x, \dot{x}) \tag{IV.3.1}$$

is a constant of motion.

As a consequence, for transformations without changing the time, the quantity (IV.3.1) reduces to

$$I(t, x(t)) = \zeta(t, x) \cdot \frac{\partial L}{\partial v}(t, x, \dot{x}). \tag{IV.3.2}$$

The well known conservation laws are those of energy momentum and angular momentum. We consider the Kepler problem to derive them using Noether's theorem,

Example IV.2. The classical Lagrangian associated Kepler problem in two dimensional case is given by

$$L(t, x, v) = \frac{1}{2}\|v\|^2 + \frac{1}{\|x\|}, \quad (x, v) \in (\mathbb{R}^*)^2 \times \mathbb{R}^2.$$

As L does not depend explicitly on time, the Lagrangian L is time translation invariant. Hence, Noether's theorem gives

$$H(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 - \frac{1}{\|x\|} = \text{constant of motion.}$$

Namely, the total energy of the system is conserved.

Obviously, the Lagrangian L is invariant under a rotation in (x, y)-plane of any angle $s \in \mathbb{R}$, because for all $R_s \in \mathrm{SO}(2)$ (the group of all orthogonal matrices) $\|R_s(x_1, x_2)^\mathsf{T}\| = \|(x_1, x_2)\|$. Noether's theorem implies that

$$\dot{x}_1 x_2 - \dot{x}_2 x_1 = \text{constant of motion.}$$

Namely, the angular momentum is conserved.

IV.4 Reminder about Hamiltonian systems

For a deeper discussion of Hamiltonian systems, we refer the reader to [4], [17], [58], [89].

Definition IV.4 (Classical Hamiltonian)**.** *The Hamiltonian is a function* $H : [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *such that for* $(q, p) \in \mathscr{C}^1\left([a, b], \mathbb{R}^d\right) \times \mathscr{C}^1\left([a, b], \mathbb{R}^d\right)$ *we have the time evolution of* (q, p) *given by the classical Hamilton's equations*

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(t, q, p) \\ \dot{p} = -\dfrac{\partial H}{\partial q}(t, q, p) \end{cases} \tag{IV.4.1}$$

This previous system can be written in matrix form by putting $z = (q, p)^\mathsf{T}$ and $\nabla H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)^\mathsf{T}$. The Hamiltonian system can be recast as

$$\dot{z} = J \cdot \nabla H,$$

where $\begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ denotes the symplectic matrix and with I_d the identity matrix on $\mathbb{R}^d$.

IV.4.1 The Legendre transform and Hamiltonian systems

The passage from the Lagrangian system to the Hamiltonian system or conversely is possible by making a suitable change of variable via the so-called *Legendre transform* (see [4, p.61], [17, p.160]). The first deals with the phase (or velocity) space whereas the other deals with the configuration (or position) space.

Based in that transform for the Lagrangian $L(q, v)$ allowing to to define a new variable called the *conjugate momentum*

$$p = \frac{\partial L}{\partial v}(t, q, v).$$

Under the assumption that the Legendre transformation is invertible, the variable v can be implicitly defined in term of (t, q, p), i.e., $v = g(t, q, p)$.

Therefore, the Euler-Lagrange equation can be rewritten equivalently as the Hamiltonian system with the corresponding Hamiltonian associated to L defined by (see [58, p.79] or [89, p.243])

$$H(t, q, p) = p \cdot g(t, q, p) - L(t, q, g(t, q, p)).$$

IV.4.2 Hamiltonian systems via variational principle

An important property of Hamiltonian systems is that their solutions correspond to *critical points* of a given functional, i.e., follow from a *variational principle*, based on the functional of the form

$$\begin{array}{rccl} \mathcal{L}_H : & \mathscr{C}^1\left([a, b], \mathbb{R}^d\right) \times \mathscr{C}^1\left([a, b], \mathbb{R}^d\right) & \longrightarrow & \mathbb{R} \\ & (q, p) & \longmapsto & \displaystyle\int_a^b L_H(t, q(t), p(t), \dot{q}(t), \dot{p}(t))dt, \end{array} \tag{IV.4.2}$$

where $L_H : [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is the Lagrangian defined by

$$L_H(t, q, p, v, w) = p \cdot v - H(t, q, p). \tag{IV.4.3}$$

Theorem IV.2. *The points* $(q, p) \in \mathscr{C}^1\left([a, b], \mathbb{R}^d\right) \times \mathscr{C}^1\left([a, b], \mathbb{R}^d\right)$ *satisfying Hamilton's equations are critical points of the functional* $\mathcal{L}_H$.

IV.5 Canonical transformations groups

In dealing with Hamiltonian systems, one need to preserve the Hamiltonian character of the equation under change of variables. As we look for the behaviour of a functional under a group action, we need to ensure that the new equation is for each element of the group again a Hamiltonian system, i.e., that we need to consider groups of canonical transformations. As a consequence of the previous paragraph, we consider transformations which are given by a *generating function*.

Definition IV.5 (Canonical group of transformations)**.** *We denote by* $\{\phi_s\}_{s\in\mathbb{R}}$ *the group of transformations given by*

$$\begin{array}{rccl} \phi_s: & [a,b]\times\mathbb{R}^d\times\mathbb{R}^d & \longrightarrow & \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \\ & (t,q,p) & \longmapsto & \big(\phi_s^0(t),\phi_s^1(q,p),\phi_s^2(q,p)\big), \end{array}$$

where

$$\begin{aligned} \phi_s^0(t) &= t + s\zeta(t) + O(s^2), \\ \phi_s^1(q,p) &= q + s\frac{\partial G}{\partial p} + O(s^2), \\ \phi_s^2(q,p) &= p - s\frac{\partial G}{\partial q} + O(s^2), \end{aligned} \tag{IV.5.1}$$

with $G(q,p)$ *is of class* $\mathscr{C}^2$.

Note that these transformation groups are projectable transformation groups.

IV.6 Hamiltonian Noether's theorem

We denote by $(q_s(\tau),p_s(\tau))$ the transform of a given solution $(q(t),p(t))$ where $\tau=\phi_s^0(t)$.

Definition IV.6 (Variational symmetry)**.** *A canonical group of transformation is a variational symmetry of* $\mathcal{L}_{H,[a,b]}$ *if for all* $s\in\mathbb{R}$ *and any subinterval* $[t_a,t_b]\subseteq[a,b]$*, we have*

$$\mathcal{L}_{H,[t_a,t_b]}(q,p) = \mathcal{L}_{H_s,[\tau_a,\tau_b]}(q_s,p_s), \tag{IV.6.1}$$

where H_s *is the Hamiltonian function associated to the system in the new variables* (q_s,p_s) *and* $[\tau_a,\tau_b]=[\phi_s^0(t_a),\phi_s^0(t_b)]$

The classical Noether's theorem for Hamiltonian systems can then be formulated as follows:

Theorem IV.3 (Noether's theorem - Hamiltonian systems)**.** *If the Hamiltonian functional possesses a variational symmetry given by* (IV.5.1) *then the quantity*

$$I = p\cdot\frac{\partial G}{\partial p} - H\zeta, \tag{IV.6.2}$$

is a first integral over the solution of the Hamiltonian systems.

We see that we recover the well known result that if the Hamiltonian is time independent then a variational symmetry is given by $G=0$ and $\zeta=1$ which leads to the fact that the Hamiltonian H itself is a first integral.

Proof. By expressing the right hand side of (IV.6.1) in terms of t, q and p allowing to obtain the variational symmetry criterion of $\mathcal{L}_{H,[a,b]}$, that is

$$\int_{t_a}^{t_b} \Big[\left(p - \partial_q G + O(s^2)\right) \frac{\left(\dot{q} + s\frac{d}{dt}(\partial_p G) + O(s^2)\right)}{1 + s\dot{\zeta} + O(s^2)} \\ - H(t + s\zeta + O(s^2), q + s\partial_p G, p - s\partial_q G + O(s^2))\Big] \left(1 + s\dot{\zeta} + O(s^2)\right) dt.$$

Now, replacing the right hand side of (IV.6.1) by the previous equality and by differentiating the equation (IV.6.1) with respect to s and taking $s = 0$, one obtain an alternative form of the invariance condition: if the Hamiltonian functional (IV.4.2) possesses a variational symmetry given by (IV.5.1) then one has

$$-\frac{\partial H}{\partial q} \cdot \frac{\partial G}{\partial p} + p \cdot \frac{d}{dt}\left(\frac{\partial G}{\partial p}\right) - \frac{\partial H}{\partial t}\zeta - H\frac{d\zeta}{dt} = 0. \tag{IV.6.3}$$

As $\dot{p} = -\partial_q H$, we have

$$-\frac{\partial H}{\partial q} \cdot \frac{\partial G}{\partial p} + p \cdot \frac{d}{dt}\left(\frac{\partial G}{\partial p}\right) = \frac{d}{dt}\left(p \cdot \frac{\partial G}{\partial p}\right), \tag{IV.6.4}$$

and the equation (IV.6.3) becomes

$$\frac{d}{dt}\left(p \cdot \frac{\partial G}{\partial p}\right) = \frac{\partial H}{\partial t}\zeta + H\frac{d\zeta}{dt}. \tag{IV.6.5}$$

Moreover as for an arbitrary function $f(q, p)$ we have

$$\frac{d}{dt}\left(P(q,p)\right) = \{H, P\} + \frac{\partial P}{\partial t}, \tag{IV.6.6}$$

over the solution of the Hamiltonian systems and where $\{\cdot,\cdot\}$ denotes the *Poisson bracket* defined for two functions $P(q, p)$ and $Q(q, p)$ by

$$\{P, Q\} = \frac{\partial P}{\partial p} \cdot \frac{\partial Q}{\partial q} - \frac{\partial Q}{\partial p} \cdot \frac{\partial P}{\partial q}. \tag{IV.6.7}$$

We deduce that the quantity in right-hand side of (IV.6.5) can then be written

$$\frac{\partial H}{\partial t}\zeta + H\frac{d\zeta}{dt} = \frac{dH}{dt}\zeta + H\frac{d\zeta}{dt} = \frac{d}{dt}\left(H\zeta\right), \tag{IV.6.8}$$

using the fact that $\{H, H\} = 0$.

The invariance condition is then equivalent to

$$\frac{d}{dt}\left[p \cdot \frac{\partial G}{\partial p} - H\zeta\right] = 0, \tag{IV.6.9}$$

and the proof is complete. □

Chapter V

Noether's Time Scales Theorems for Lagrangian Systems

We prove a time scales version of the Noether theorem relating group of symmetries and conservation laws in the framework of the shifted and nonshifted Δ-calculus of variations. Our result extends the continuous version of the Noether theorem as well as the discrete one and corrects a previous statement of Bartosiewicz and Torres in [8] which implies also that the second Euler-Lagrange equation on time scales as derived in [7] is incorrect. Using the Caputo duality principle introduced in [19], we provide the corresponding Noether theorem on time scales in the framework of the shifted and nonshifted ∇-calculus of variations.

This Chapter is based on the published article "Noether's-type theorems on time scales" *with B. Anerot, J. Cresson, and F. Pierret, Journal of Mathematical Physics, 2020.*

V.1 Introduction and statement of the problem

In 2004, the time scales theory was used by Bohner [10] and Hilscher and Zeidan [53] to develop a calculus of variations on time scales. This first attempt was then followed by numerous generalizations. In this chapter, we focus on two specific settings, namely, the shifted calculus of variations as introduced in [10] and the nonshifted one as considered in [37] (see also [14]).

In this context, many natural problems arise. One of them is to generalize to the time scales setting classical results of the calculus of variation in the continuous case. One of these problems is to obtain a time scale analog of the Noether theorem relating group of symmetries and conservation laws [4].

The derivation of Noether's theorem on the time scales calculus that considered by Z. Bartosiewicz and D.F.M. Torres in [8] was the beginning of problem and then in [7]. Two different strategies of proof are used:

- First, they proposed in [8] to derive the Noether theorem for transformations depending on time from the easier result obtained for transformations without changing the time. In [32], we call *Jost's method* this way of proving the Noether theorem as a classical reference is contained in the book [58].

- Another method is proposed in [7], where a *second Euler–Lagrange equation* is derived [7, Theorem 5, p.12] and from which the Noether theorem is deduced (see [7, Section 4, Theorem 6]). The so-called *second Euler-Lagrange equation* obtained in [7] is given by

$$\Delta\left[\mathscr{H}(\cdot, x^\sigma, \Delta x)\right](t) + \frac{\partial L}{\partial t}(t, x^\sigma, \Delta x) = 0, \tag{$\mathrm{EL}^{2\mathrm{nd}}_\sigma$}$$

where

$$\mathscr{H}(t,x,v) = -L(t,x,v) + v\frac{\partial L}{\partial v}(t,x,v) + \mu(t)\frac{\partial L}{\partial t}(t,x,v)$$

which is satisfied over the solutions of the shifted Euler-Lagrange equation for all $t \in \mathbb{T}^{\kappa^2}$ and can be easily implemented.

However, generalizing the Jost method of proof outside of the classical calculus of variations need to be done carefully as many ingredients in the proof are intimately related to properties of the differential calculus (see [32, Section 4.2]). By the way, previous attempts to apply this strategy in different contexts (see for example [40]) have leaded to false results (see [32], [38]).

Having these problems in mind, we decide to check numerically first on an example provided in [8] the validity of the main result of [8, Theorem 4] and [7, Theorem 5].

Precisely, we use Example 3 of [8], p.1226 defined as follows (more details will be given in Section V.5.1). We consider the Lagrangian introduced in [8] and defined by

$$L(t,x,v) = \frac{x^2}{t} + tv^2 \tag{V.1.1}$$

for $t \in \mathbb{R} \setminus \{0\}$ and $(x,v) \in \mathbb{R}^2$. In [8], the authors consider the time scales

$$\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}. \tag{V.1.2}$$

In that case, $\sigma(t) = 2t$ for all $t \in \mathbb{T}$ and $\Delta\sigma(t) = 2$. The shifted Euler–Lagrange equation associated with L is given by

$$\Delta\big[t\Delta x(t)\big] = \frac{x^\sigma}{t}, \tag{V.1.3}$$

One can prove that the shifted Lagrangian functional associated to L defined by (see [8, Eq. (4), p.1222]):

$$\mathcal{L}^\sigma_{\Delta,\mathbb{T}}(x) = \int_a^b L\left(t, x^\sigma(t), \Delta x(t)\right)\Delta t. \tag{V.1.4}$$

is invariant in the sense of Definition 5 in [8, p.1224] under the following group of transformations (see [8, p.1226]) and Section V.5.1.2):

$$\{g_s(t,x) = (te^s, x)\}_{s\in\mathbb{R}}. \tag{V.1.5}$$

All the assumptions of the shifted time scales Noether theorem given in [8, Theorem 4], are satisfied and as a consequence the following quantity

$$C(t, x^\sigma, v) = 2t\left(\frac{(x^\sigma)^2}{t} - tv^2\right), \tag{V.1.6}$$

must be a constant of motion (see [8, Example 3, p.1226]) in the sense of Definition 4 in [8].

We then test numerically if the function C is constant over the solutions of the Euler–Lagrange equation and at the same time if the right hand side of equation (EL_σ^{2nd}) is equal to zero.

In the following, we plot the solution x of the shifted Euler-Lagrange equation (in red) for $t = 0, 2, 4, 8, 16, 32$ and with initial conditions $x(0) = 0$ and $\Delta(x)(0) = 0.1$. We then plot the quantities C (in blue) and the left-hand side of (EL_σ^{2nd}) (in green) over the solutions of equation (V.1.1). The simulations give the following results:

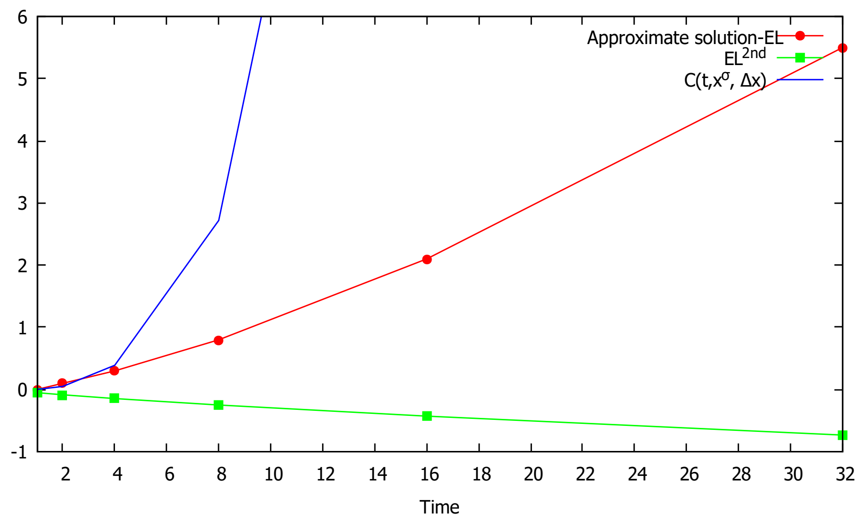


Figure V.1: $x_0 = 0, \Delta x_0 = 0.1$.

These simulations clearly show that the function C is not a constant of motion and that the second Euler–Lagrange equation is not satisfied providing a counter example to these two results. It must be noted that this invalidates many other results which use the former results (see for example [90, Theorem 3, p.5], where the second order Euler–Lagrange equation is used in the proof (see Eq. (33) in [90]).

The aim of this chapter is to derive a time scales version of the Noether theorem in the shifted and nonshifted calculus of variations settings. We provide two different proofs:

- First, we follows the initial strategy used by Z. Bartosiewicz and D.F.M. Torres in [8] which refers to a time scales analogue of a classical proof exposed by J. Jost and X. Li-Jost in [58]. We point out several difficulties which are in fact inherent to the Jost's method (see [32]). This first proof is not the simplest one but it explains where and why the initial proof given in [8] is not correct.

- Second, a more classical one which can be called "*direct*", which consists in deriving the invariance relation with respect to the parameter of the transformation group and manipulating the obtained expression in order to provide a constant of motion. Although less elegant than the previous one, it is the most easiest one.

Organization of the chapter. In Section VI.4 we state the Noether theorem on time scales in the context of the Δ shifted or nonshifted calculus of variations. In particular, Section VI.3 precises the notion of transformation groups in the context of time scales calculus and those of admissible projectable transformations groups. Section V.3 gives the proof of our main result using the Jost method. The proofs of several technical Lemmas are postponed in Section V.8. Section V.4 gives alternative proofs of our results with a so called direct method. In Section VI.6, we discuss several examples and provide numerical simulations. We first study an example given by Bartosiewicz and Torres in [8]. We then discuss results obtained in the same context by X.H. Zhai and L.Y. Zhang in [90] about a time scales version of the Kepler problem in the plane. Here again, we prove that the results presented in [90] are not correct. In Section V.6, we use the *Caputo duality principle* in time scales as presented in [19] to obtain the Noether theorem on time scales for the ∇ shifted and nonshifted calculus of variations. Our result differs also from the one obtained by N. Martins and D.F. Torres in [75]. We discuss an example proposed by X.H. Zhai and L.Y. Zhang in [90] and prove that the result of [75] are indeed incorrect.

V.2 Main results

In the following, $\mathbb{T}$ denotes a bounded time scales with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$ and we assume that $\mathrm{card}(\mathbb{T}) \geq 3$ ensuring that $\mathbb{T}^\kappa_\kappa \neq \emptyset$.

V.2.1 Admissible transformations group

We consider a special class of symmetry groups of differential equations called *projectable* or *fiber-preserving* (see [83, p.93]) and given by

$$\begin{array}{rccc} g_s: & [a,b]\times\mathbb{R}^n & \longrightarrow & \mathbb{R}\times\mathbb{R}^n \\ & (t,x) & \longmapsto & (g_s^0(t), g_s^1(x)) \end{array} \tag{V.2.1}$$

where $\{g_s\}_{s\in\mathbb{R}}$ is a one parameter group of diffeomorphisms satisfying $g_0 = \mathrm{Id}$. The associated *infinitesimal (or local) group action* (see [83, p.51]) or transformations is obtained by making a Taylor expansion of g_s around $s = 0$:

$$g_s(t,x) = g_0(t,x) + s\,\frac{\partial g_s(t,x)}{\partial s}\bigg|_{s=0} + O(s). \tag{V.2.2}$$

The *transform* (see [83, p.90]) of a given function $x(t)$ identified with its graph $\Gamma_x = \{(t, x(t)),\ t \in [a,b]\}$ by g_s is easily obtained introducing a new variable τ defined by $\tau = g_s^0(t)$. The transform of x denoted by $\tilde{x}$ is then given by

$$\tau \longrightarrow (\tau, \bar{x}(\tau)) = (\tau, g_s^1 \circ x \circ (g_s^0)^{-1}(\tau)).$$

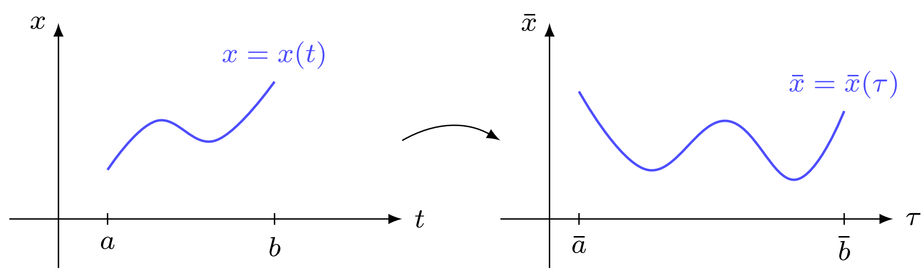


Figure V.2: The transform of a function x by g_s.

Remark V.1. *In general, the transform of a given function is not so easy to determine explicitly (see [83], Example 2.21, p.90-91) and one must use the implicit function theorem in order to recover the transform of x. This is precisely the reason why we restrict our attention to projectable or fiber-preserving symmetry groups.*

Working with time scales imposes some restrictions on the transformation groups that one can consider. In the following, we need the notion of $(\Delta, \mathbb{T})$**-admissible projectable group of transformations**:

Definition V.1 ($(\Delta,\mathbb{T})$-admissible projectable group of transformations)**.** *A projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$ is called a $(\Delta,\mathbb{T})$-admissible projectable group of transformations if for all $s \in \mathbb{R}$, the function g_s^0 verifies:*

- *g_s^0 is strictly increasing,*
- *$\Delta g_s^0 \neq 0$ and Δg_s^0 is rd-continuous and such that,*
- *the set defined by $\widetilde{\mathbb{T}}_s = g_s^0(\mathbb{T})$ is a time scales.*
- *$\Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^0\right)^{-1}$ exists.*

V.2.2 Noether's theorem on time scales in the nonshifted calculus of variations

Let L be a Lagrangian function. We can associate to L a functional $\mathcal{L}_{\Delta,\mathbb{T}} : \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}_{\Delta,\mathbb{T}}(x) = \int_a^b L(t, x(t), \Delta x(t))\Delta t, \tag{V.2.3}$$

called the *nonshifted Lagrangian functional* over the time scales $\mathbb{T}$.

If σ is ∇-differentiable on $\mathbb{T}_\kappa$, then the critical points of $\mathcal{L}_{\Delta,\mathbb{T}}$ are solutions of the $\nabla\circ\Delta$-*differential Euler–Lagrange equation* (see [14, Theorem 1, p.548]):

$$\nabla\left[\frac{\partial L}{\partial v}(\cdot, x, \Delta x)\right](t) = \nabla\sigma(t)\frac{\partial L}{\partial x}(t, x(t), \Delta x(t)), \tag{EL$^{\nabla\circ\Delta}$}$$

for every $t \in \mathbb{T}^\kappa_\kappa$.

V.2.2.1 Invariance of functionals and variational symmetries

We have the following time scales generalization of the definition of a *variational symmetry group* of a *nonshifted Lagrangian functional on time scales* (see [83, Definition 4.10, p.253]):

Definition V.2 (Variational symmetries)**.** *The $(\Delta,\mathbb{T})$-admissible group of transformations $\{g_s\}_{s\in\mathbb{R}}$ is a variational symmetry group of the nonshifted functional (V.2.3) if whenever $I = [t_a, t_b]$ is a subinterval of $[a, b]$ with $t_a, t_b \in \mathbb{T}$ and $x \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ such that its transform under g_s denoted by $\tilde{x}$ is defined over $\tilde{I}_s = [\tau_a, \tau_b]$ which is a subset of $g_s^0([a, b]) = [\tilde{a}_s, \tilde{b}_s]$, then*

$$\mathcal{L}_{\Delta,\mathbb{T}}(x) = \mathcal{L}_{\widetilde{\Delta},\widetilde{\mathbb{T}}_s}(\widetilde{x}). \tag{V.2.4}$$

It is interesting to give an explicit formulation of this definition. Indeed, according to definition of the functional (V.2.3), we can write (V.2.4) as

$$\int_{t_a}^{t_b} L\left(t, x(t), \Delta x(t)\right)\Delta t = \int_{\tau_a}^{\tau_b} L_s\left(\tau, g_s^1\circ x\circ (g_s^0)^{-1}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1\circ x\circ (g_s^0)^{-1}\right)(\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau \tag{V.2.5}$$

where $\tau_a = g_s^0(t_a)$ and $\tau_b = g_s^0(t_b)$.

V.2.2.2 Noether's Theorem on time scales - nonshifted case

Our main result is the following nonshifted version of the Noether theorem on time scales:

Theorem V.1 (Noether's theorem - Nonshifted case). *Let $\mathbb{T}$ be a time scales such that σ is ∇-differentiable on $\mathbb{T}_\kappa$ and $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ be a $(\Delta,\mathbb{T})$-admissible projectable group of transformations which is a variational symmetry of the nonshifted Lagrangian functional on time scales $\mathbb{T}$ given by*

$$\mathcal{L}_{\Delta,\mathbb{T}}(x) = \int_a^b L\left(t, x(t), \Delta x(t)\right) \Delta t$$

and

$$\mathbf{X} = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}, \tag{V.2.6}$$

be the infinitesimal generator of G. Then, the function

$$I(t,x,v) = -\zeta^\sigma(t)H(\star) + \xi^\sigma(x)\cdot\partial_v L(\star) + \int_a^t \zeta\Big[\nabla\sigma\partial_t L(\star) + \nabla\big(H(\star)\big)\Big]\nabla t, \tag{V.2.7}$$

where $H : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is defined by

$$H(t,x,v) = -L(t,x,v) + \partial_v L(t,x,v)\cdot v, \tag{V.2.8}$$

and $(\star) = (t, x(t), \Delta x(t))$, is a constant of motion over the solution of the time scales Euler–Lagrange equation ($\mathrm{EL}^{\nabla\circ\Delta}$), i.e., that

$$\nabla\big[I\left(\cdot\,, x(\cdot)\right)\big](t) = 0, \tag{V.2.9}$$

for all solutions x of the time scales Euler–Lagrange equations and any $t \in \mathbb{T}_\kappa^\kappa$.

The proof is given in Section V.3.

In the continuous case $\mathbb{T} = [a,b]$, one obtains the classical form of the integral of motion

$$I(t,x) = -\zeta(t)H(t,x,\dot{x}) + \xi(x)\cdot\partial_v L(t,x,\dot{x}). \tag{V.2.10}$$

Indeed, if $\mathbb{T} = [a,b]$ then σ is ∇-differentiable on $\mathbb{T}_\kappa$ with $\nabla[\sigma] = 1$ and moreover, on the solutions of the Euler–Lagrange equation one has the identity

$$-\frac{\partial L}{\partial t}(t,x,\dot{x}) = \frac{d}{dt}\left(H(t,x,\dot{x})\right) \tag{V.2.11}$$

which is called the *second Euler–Lagrange equation* [89].

In the discrete case, $\mathbb{T} = \mathbb{Z}$ and transformations without changing time, one recovers the classical integral (see [15, Theorem 12, p.885] and also [45]):

$$I(x) = \xi^\sigma(x)\cdot\frac{\partial L}{\partial v}(t,x,\Delta x). \tag{V.2.12}$$

V.2.3 Noether's Theorem on time scales in the shifted calculus of variations

Let L be a Lagrangian function. We consider the functional $\mathcal{L}^{\sigma}_{\Delta,\mathbb{T}}(x)$ defined for all $x \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ by

$$\mathcal{L}^{\sigma}_{\Delta,\mathbb{T}}(x) = \int_a^b L\left(t, x^{\sigma}(t), \Delta x(t)\right) \Delta t. \tag{V.2.13}$$

Let $\mathbb{T}$ be a time scale such that σ is Δ-differentiable on $\mathbb{T}^{\kappa}$. The critical points of $\mathcal{L}^{\sigma}_{\Delta,\mathbb{T}}$ are solutions of the **shifted time scales Euler–Lagrange equation** given by (see [10, Theorem 4.2, p.344])

$$\Delta\left[\frac{\partial L}{\partial v}(\cdot, x^{\sigma}, \Delta x)\right](t) = \frac{\partial L}{\partial x}(t, x^{\sigma}(t), \Delta x(t)), \tag{$\mathrm{EL}^{\Delta\circ\Delta}$}$$

for every $t \in \mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$.

Remark V.2 (A remark on the shifted calculus of variations)**.** *Although the shifted calculus of variations was introduced first in the literature, the definition of the functional (V.2.13) seems to be non-natural with respect to a discretisation procedure of the continuous Lagrangian functional and in fact leads to very bad numerical integrator of the continuous equation. This is due to the fact that in this case, the second order derivative d^2/dt^2 is approximated by $\Delta \circ \Delta$ which is an operator of order one with respect to the time step used as a discretization step, instead of order 2 for the $\nabla \circ \Delta$ operator which appears in the non-shifted case.*

However, leaving this aspect, one can justify the use of the shifted calculus of variations as follows: Going back to I. Newton's seminal work Philosophiae Naturalis Principia Mathematica published first in 1866 (a reprint can be found in [49] with other texts of interest), we can take a look at the first place were he derived the now famous law of motions for a body under the gravitational force. We refer to the discussion given by R. Feynman in [39] for more details.

He explains that the motion of a body around a massive body with an initial speed v_0 evolves during a short amount of time $t_1 - t_0 = h$ following the inertia principle introduced by Galileo. The particle then follows a straight line between the initial position x_0 and $\tilde{x_1}$ whose length is given by $v_0 h$. However, at time t_1, the effect of the force F during the time h is taken into account and assumed to be of magnitude $F(x_0)h^2$. This reasoning is illustrated by I. Newton in his book by the following picture (see [49, p.431] and also [39, p.84]):

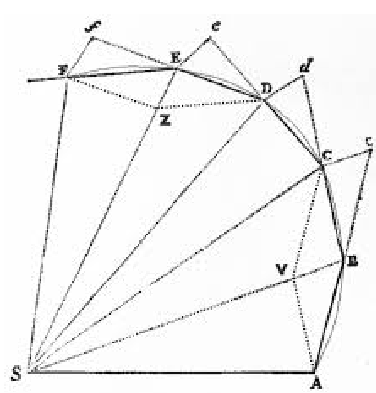


Figure V.3: Newton's illustration for the motion of a planet around a star

As the force is assumed to be directed toward the massive body, I. Newton deduces that the position of the particle at time t_1 satisfies $x_1 - \tilde{x_1} = F(x_0)h^2$, which leads to $x_1 - x_0 = v_0 h + F(x_0)h^2$ and finally, denoting $\Delta x(t_1) = (x_1 - x_0)/h$ and $\Delta x(t_0) = v_0$, to the equation

$$\Delta x(t_1) - \Delta x(t_0) = F(x_0)h, \tag{V.2.14}$$

and to the classical writing of Newton's fundamental law of motion

$$\Delta(\Delta x)(t_0) = F(x_0). \tag{V.2.15}$$

As a consequence, I. Newton's first derivation of the law of motion leads to an equation where only the Δ derivative appears. This equation can only be recovered using the shifted calculus of variations.

The notion of invariance is adapted to the shifted case as follows:

Definition V.3 (Shifted invariance)**.** *A time scales Lagrangian functional $\mathcal{L}^{\sigma}_{\Delta,\mathbb{T}}$ is said to be invariant under a $(\Delta,\mathbb{T})$-admissible projectable group of transformations $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ if and only if for any subinterval $[t_a,t_b] \subset [a,b]$ with $t_a, t_b \in \mathbb{T}$, for any $s \in \mathbb{R}$ and $x \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$*

$$\int_{t_a}^{t_b} L\left(t, x^{\sigma}(t), \Delta x(t)\right)\Delta t \;=\; \int_{\tau_a}^{\tau_b} L_s\left(\tau, \left[g_s^1 \circ x \circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1 \circ x \circ (g_s^0)^{-1}\right](\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau \tag{V.2.16}$$

where $\tau_a = g_s^0(t_a)$ and $\tau_b = g_s^0(t_b)$, $\widetilde{\mathbb{T}}_s = g_s^0(\mathbb{T})$ and $\tilde{\sigma}_s$ is the forward jump operator over $\widetilde{\mathbb{T}}_s$.

Theorem V.2 (Noether's theorem - σ-shifted case)**.** *Let $\mathbb{T}$ be a time scale such that σ is Δ-differentiable on $\mathbb{T}^{\kappa}$. Let $G = \{g_s(t,x) = (g_s^0(t), g_s^1(x))\}_{s\in\mathbb{R}}$ be a $(\Delta,\mathbb{T})$-variational symmetry of $\mathcal{L}^{\sigma}_{\Delta,\mathbb{T}}$ with the corresponding infinitesimal generator given by*

$$\mathbf{X} = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}. \tag{V.2.17}$$

Then, the quantity

$$I(t, x^{\sigma}, v) = -\mathscr{H}(\star_{\sigma})\zeta(t) + \partial_v L(\star_{\sigma})\xi(x) + \int_a^t \zeta^{\sigma}(t)\Big(\Delta\big[\mathscr{H}(\star_{\sigma})\big] + \partial_t L(\star_{\sigma})\Big)\Delta t \tag{V.2.18}$$

where $(\star_{\sigma}) = (t, x^{\sigma}(t), \Delta x(t))$ and $\mathscr{H}$ is given by

$$\mathscr{H}(t,u,v) = -L(t,x,v) + \partial_v L(t,x,v)\cdot v + \partial_t L(t,u,v)\mu(t), \tag{V.2.19}$$

is a constant of motion over the solution of the time scales Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$), i.e., that

$$\Delta\big[I(\cdot, x^{\sigma}, \Delta x)\big](t) = 0, \tag{V.2.20}$$

for all solutions x of ($\mathrm{EL}^{\Delta\circ\Delta}$) and any $t \in \mathbb{T}^{\kappa^2}$.

In the continuous case $\mathbb{T} = [a,b]$, we have $\sigma(t) = t$ and $\mu(t) = 0$, so that one obtains the classical form of the integral of motion (V.2.10).

V.2.4 Comparison with the Noether theorem on time scales obtained by Z. Bartosiewicz and D.F.M Torres

In [8], Z. Bartosiewicz and D.F.M. Torres prove a Noether's theorem on time scale which leads to the statement that the quantity

$$C\left(t, x^{\sigma}, v\right) = -\mathscr{H}(\star_{\sigma})\zeta(t,x) + \partial_v L(\star_{\sigma}) \cdot \xi(t,x) \tag{V.2.21}$$

is a constant of motion over the solutions of ($\mathrm{EL}^{\Delta\circ\Delta}$).

As we can see, we have an extra term in our result given by

$$\int_a^t \zeta^{\sigma}(t)\Big(\Delta\big[\mathscr{H}(\star_{\sigma})\big] + \partial_t L(\star_{\sigma})\Big)\Delta t.$$

The difference comes from the fact that Z. Bartosiewicz and D.F.M. Torres [8] *implicitly* assume that the following equation

$$\Delta\big[\mathscr{H}(\cdot, x^{\sigma}, \Delta x)\big](t) = -\frac{\partial L}{\partial t}(t, x^{\sigma}(t), \Delta x(t)), \tag{$\mathrm{EL}_{\sigma}^{\mathrm{2nd}}$}$$

called the second order Euler-Lagrange equation is satisfied over the solutions of the shifted Euler-Lagrange equation. The previous equation does not appear in [8] but follow from one implicit assumption made by the authors during the derivation of their result using the Jost method of proof: Informally, the authors construct what is called an extended Lagrangian enabling them to formulate the invariance property for transformations with time of the initial Lagrangian functional to an invariance property for a transformation without transforming time for the extended Lagrangian functional. Doing this, the idea is to apply the Noether theorem without transforming time to the extended formulation. However, in order to do so, one must prove that solutions of the initial Euler-Lagrange equation produce solutions of the Euler-Lagrange equation for the extended Lagrangian. This point was not discussed in [8]. The second Euler-Lagrange equation corresponds exactly to the condition one needs to impose on the solutions of the initial Euler-Lagrange equation in order that they corresponds to solutions of the extended Euler-Lagrange equation. We refer to Section V.3.2 for more details and in particular Lemma V.9.

As a consequence, our result coincides with the one of Z. Bartosiewicz and D.F.M. Torres [8] if and only if the second order Euler-Lagrange equation on time scales is valid. However, as already showed in the introduction by the simulations on an explicit example, this is not true. In the following, we give a counter-example to the second order Euler-Lagrange equation where all computations can be made explicitly.

V.2.4.1 Explicit counter-example to the second order Euler-Lagrange equation on time scales

Let us consider the Lagrangian

$$L(x^{\sigma}, \Delta x) = (\Delta x)^2 + 4x^{\sigma}. \tag{V.2.22}$$

The shifted Euler-Lagrange equation is given by

$$\Delta\left[\Delta x\right] = 2. \tag{V.2.23}$$

As $\partial_t L = 0$, the quantity $\mathscr{H}$ reduces to

$$\mathscr{H}(x^\sigma, \Delta x) = (\Delta x)^2 - 4x^\sigma. \tag{V.2.24}$$

We have the following Lemma:

Lemma V.1. *The function $\Delta\mathscr{H}$ is equal to*

$$\Delta\mathscr{H} = 4\left[-\mu - 2\mu\Delta\mu - \Delta x \Delta\mu\right], \tag{V.2.25}$$

over the solutions of the shifted Euler-Lagrange equation.

Proof. We have (see [11],1.36 p.337) that for any function $u \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}$ such that $\Delta(u^\sigma)$ exists, the relation

$$\Delta(u^\sigma) = (1 + \Delta\mu)(\Delta u)^\sigma. \tag{V.2.26}$$

Moreover, using the Leibniz formula we have

$$\begin{aligned}\Delta\left((\Delta x)^2\right) &= \Delta(\Delta x)\Delta x + (\Delta x)^\sigma \Delta(\Delta x),\\ &= \Delta(\Delta x)\left(\Delta x + (\Delta x)^\sigma\right).\end{aligned} \tag{V.2.27}$$

As a consequence, we obtain

$$\begin{aligned}\Delta\mathscr{H} &= \Delta\left((\Delta x)^2\right) - 4\Delta(x^\sigma),\\ &= \Delta(\Delta x)\left(\Delta x + (\Delta x)^\sigma\right) - 4(1+\Delta\mu)(\Delta x)^\sigma.\end{aligned} \tag{V.2.28}$$

Using the shifted Euler-Lagrange equation, one obtain

$$\Delta\mathscr{H} = 2\left(\Delta x + (\Delta x)^\sigma\right) - 4(1+\Delta\mu)(\Delta x)^\sigma. \tag{V.2.29}$$

Moreover, we have the classical relation for $u \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}$ (see [11],(iv),p.6):

$$u^\sigma = u + \mu\Delta u, \tag{V.2.30}$$

which gives

$$(\Delta x)^\sigma = \Delta x + \mu\Delta(\Delta x) = \Delta x + 2\mu, \tag{V.2.31}$$

thanks to the shifted Euler-Lagrange equation.

As a consequence, replacing in the expression of $\Delta\mathscr{H}$, one obtain

$$\begin{aligned}\Delta\mathscr{H} &= 2\left(2\Delta x + 2\mu\right) - 4(1+\Delta\mu)(\Delta x + 2\mu),\\ &= 4\left[-\mu - 2\mu\Delta\mu - \Delta x\Delta\mu\right],\end{aligned} \tag{V.2.32}$$

which concludes the proof. □

As a consequence, any time scales such that μ is a non zero constant lead to a counter example to the second order Euler-Lagrange equation. In particular, we have

Lemma V.2. *Let $\mathbb{T} = \mathbb{Z}$, then $\Delta\mathscr{H} = -4$.*

Proof. For $\mathbb{T} = \mathbb{Z}$, we have $\mu = 1$ for all $t \in \mathbb{T}$. As a consequence, we have $\Delta\mu = 0$. Replacing in the formula (V.2.25), we obtain $\Delta\mathscr{H} = -4$. □

V.2.4.2 Connection with energy preserving variational integrators

We can go further relying on the fact that for uniform time scales, the shifted Euler-Lagrange equation can be interpreted as a *variational integrator* (see [45], [74]).

Assuming that $\mathbb{T}$ is the uniform time scale over $[a, b]$, i.e., that $\mathbb{T} = \{t_i = a + ih,\ i = 0, \ldots, N\}$ with $h = (b - a)/N$. Then $\mu(t) = h$ for all $t \in \mathbb{T}^\kappa$. If the Lagrangian L is independent of the time variable, then $\partial_t L = 0$ and the quantity ($\mathrm{EL}_\sigma^{2\mathrm{nd}}$) reduced to

$$\Delta\big[\mathscr{H}(\cdot, x^\sigma(\cdot), \Delta x(\cdot))\big](t) = 0, \quad \forall\, t \in \mathbb{T}^\kappa. \tag{V.2.33}$$

The quantity $\mathscr{H}$ corresponds to the Hamiltonian associated to the Lagrangian systems and its value to the *energy* of the system. However, it is well known since the work of Z. Ge and J.E. Marsden [91] that ”fixed time step variational integrators derived from the discrete variational principle cannot preserve the energy of the system exactly”. This implies precisely that the time scales second Euler-Lagrange equation is not valid in full generality.

We refer to the book of E. Hairer, C. Lubich and G. Wanner *Geometric numerical integration* [45] for more details, in particular Chapter VI.6 about variational integrators and Chapter IX.8 for a a discussion of long-term energy conservation of symplectic numerical schemes.

Remark V.3. *In [73], A.B. Malinowska and N. Martins discuss in full generality the derivation of a second Noether Theorem on time scales. In ([73, Remark 23, p.8]) they recover the second Euler-Lagrange equation derived in [7] as a special case. As a consequence, the previous discussion invalidate also the results proved in [73].*

V.3 Proof of the main results using the Jost method

The terminology of *Jost's method* was introduced in [32] to designate a particular way of proving the classical Noether theorem which can be found in [58]. The idea is very simple and elegant. One extend the set of variables, incorporating the time variable, in order to see the invariance of the functional under a symmetry group with transformation in time as an invariance of a new functional but for a symmetry group without transformation in the new "time" variable. The idea being then to apply the well known Noether theorem in this case to obtain the desired constant of motion. In [32], we have identified several steps in the method:

- First, rewrite the invariance condition in order to have an equality between two integrals over the same domain.
- The first step leads to the introduction of an *extended Lagrangian* and a new set of paths.
- Rewrite the initial invariance condition with transformation in time as an invariance condition for the extended Lagrangian for a transformation without transforming "time".
- Look for the correspondence between the solution of the initial Euler-Lagrange equation and the Euler-Lagrange equation associated to the extended Lagrangian.
- Apply the invariance characterization and derive a constant of motion.

The first three steps impose some specific constraints in the time scales framework due to the fact that the chain rule formula and the substitution formula are not always valid. However, the main problem comes from the Euler-Lagrange equation satisfied by the extended Lagrangian. Although this equation is always satisfied by solution of the initial Euler-Lagrange equation in the continuous case, this implication is no longer valid in general for an arbitrary time scales. This is precisely where some arguments given in [8] are incomplete. The end of the computations are only technical.

V.3.1 The nonshifted case

V.3.1.1 Rewriting the invariance condition and the extended Lagrangian

We first rewrite the invariance relation (V.2.5) in order to have the same domain of integration.

Lemma V.3. *Let G be a $(\Delta,\mathbb{T})$-variational symmetry of the nonshifted time scales Lagrangian functional $\mathcal{L}_{\Delta,\mathbb{T}}$, then, we have*

$$\int_a^b L\left(t,x(t),\Delta x(t)\right)\Delta t=\int_a^b L_s\left(g_s^0(t),(g_s^1\circ x)(t),\Delta\left(g_s^1\circ x\right)(t)\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t. \qquad \text{(V.3.1)}$$

The proof is given in Section V.8.1.

As for the classical case, we construct an extended Lagrangian functional which enables us to rewrite the invariance condition for a transformation group changing time as the invariance of a new functional under a transformation group without changing time.

Let us denote by $\mathbb{L}:\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^*\times\mathbb{R}^d\longrightarrow\mathbb{R}$ the Lagrangian function defined by

$$\mathbb{L}(t,x,w,v)=L\left(t,x,\frac{v}{w}\right)w. \qquad \text{(V.3.2)}$$

which is the same as the classical case and called the **extended Lagrangian**.

We denote by $\mathcal{L}_{\mathbb{L}}(t,x)$ the nonshifted Lagrangian functional associated to $\mathbb{L}$ defined for all $t\in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ strictly increasing and $x\in\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ such that $\Delta_{\widetilde{\mathbb{T}}}(x\circ t)$ exists where $\widetilde{\mathbb{T}}=t(\mathbb{T})$ by

$$\mathcal{L}_{\mathbb{L}}(t,x)=\int_a^b \mathbb{L}\left(t(\tau),(x\circ t)(\tau)),\Delta[t](\tau),\Delta[x\circ t](\tau))\right)\Delta\tau, \qquad \text{(V.3.3)}$$

is called the **nonshifted extended Lagrangian functional**.

We define the *time scales bundle path class* denoted by $\mathcal{F}$ and defined by

$$\mathcal{F}=\{(t,x)\in\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})\times\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})\ ;\ \tau\longmapsto(t(\tau),(x\circ t)(\tau)))=(\tau,x(\tau))\}. \qquad \text{(V.3.4)}$$

We have the following proposition:

Proposition V.1. *The restriction of the Lagrangian function $\mathcal{L}_{\mathbb{L}}$ to a path $\gamma=(t,x)\in\mathcal{F}$ satisfies*

$$\mathcal{L}_{\mathbb{L}}(t,x)=\mathcal{L}_{\Delta,\mathbb{T}}(x). \qquad \text{(V.3.5)}$$

Proof. Let $\gamma = (t, x) \in \mathcal{F}$. By definition, we have

$$\mathbb{L}\left(t(\tau), x(\tau), \Delta[t](\tau), \Delta[x \circ t](\tau)\right) = L\left(t(\tau), (x \circ t)(\tau)), \Delta[x \circ t](\tau))\frac{1}{\Delta[t](\tau)}\right)\Delta[t](\tau). \tag{V.3.6}$$

As γ is a bundle path, we have $t(\tau) = \tau$ and $\Delta[t](\tau) = 1$. As t is strictly increasing, $t \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ and $x \circ t = x$ belongs to $\mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$, the functional (V.3.3) is well defined and we obtain

$$\begin{aligned} \mathcal{L}_{\mathbb{L}}(t, x) &= \int_a^b \mathbb{L}\left(t(\tau), (x \circ t)(\tau), \Delta[t](\tau), \Delta[x \circ t](\tau))\right)\Delta\tau, \\ &= \int_a^b L\left(\tau, x(\tau), \Delta x(\tau)\right)\Delta\tau = \mathcal{L}_{\Delta,\mathbb{T}}(x), \end{aligned} \tag{V.3.7}$$

which concludes the proof. □

V.3.1.2 Invariance of the extended Lagrangian

We now reformulate the initial existence of a variational symmetry for $\mathcal{L}_{\Delta,\mathbb{T}}$ under the group G as an invariance of the extended Lagrangian:

Lemma V.4. *Let $\mathcal{L}_{\Delta,\mathbb{T}}$ be a time scales Lagrangian functional invariant under the $(\Delta,\mathbb{T})$-admissible projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$. Then, the time scales Lagrangian functional $\mathcal{L}_{\mathbb{L}}$ is invariant over $\mathcal{F}$ under the $(\Delta,\mathbb{T})$-admissible projectable group of transformations $\{g_s\}_{s\in\mathbb{R}}$.*

The proof is given in Section V.8.3.

In order to apply the Noether theorem for transformations without changing time, one needs to check that the solutions of the time scales Euler-Lagrange equation produce solutions of the extended Lagrangian systems.

Lemma V.5. *A path $\gamma = (t, x) \in \mathcal{F}$ is a critical point of $\mathcal{L}_{\mathbb{L}}$ if, and only if, x is a critical point of $\mathcal{L}_{\Delta,\mathbb{T}}$ and for all $t \in \mathbb{T}^{\kappa}_{\kappa}$ we have*

$$\nabla\sigma(t)\frac{\partial L}{\partial t}(t, x(t), \Delta x(t)) = \nabla\left[L(\cdot, x, \Delta x) - \Delta x\frac{\partial L}{\partial v}(\cdot, x, \Delta x)\right](t). \tag{EL$^{\text{2nd}}$}$$

The proof is given in Section V.8.2.

Contrary to the continuous case, Lemma V.5 implies that extended solutions of the initial Lagrangian are not automatically solutions of the extended Euler–Lagrange equation. This implies that one can not use the Noether theorem proved in [14, Theorem 2, p.553] but only the infinitesimal invariance criterion as formulated in [14, Eq. (32), p.553].

V.3.1.3 Proof of the nonshifted time scales Noether Theorem

We deduce from Lemma V.4 and the *necessary condition of invariance* given in ([14], equation (32) p.553) that

$$\partial_t L(\star)\zeta + \partial_x L(\star)\cdot\xi + \partial_v L(\star)\cdot\Delta\xi + \left[L(\star) - \partial_v L(\star)\cdot\Delta x\right]\Delta\zeta = 0. \tag{V.3.8}$$

Multiplying equation (V.3.8) by $\nabla\sigma$ and using the Time scales Euler–Lagrange equation (EL$^{\nabla\circ\Delta}$), we obtain

$$\nabla\sigma\partial_t L(\star)\ \zeta + \nabla\sigma\partial_v L(\star)\cdot\Delta\xi + \nabla\left[\partial_v L(\star)\right]\cdot\xi + \nabla\sigma\left[L(\star) - \partial_v L(\star)\cdot\Delta x\right]\ \Delta\zeta = 0. \tag{V.3.9}$$

Using the Leibniz formula (I.2.5), we have

$$\nabla\sigma\partial_t L(\star)\ \zeta + \nabla\left[\partial_v L(\star)\cdot\xi^\sigma\right] + \nabla\sigma\left[L(\star) - \partial_v L(\star)\cdot\Delta x\right]\ \Delta\zeta = 0. \tag{V.3.10}$$

Trying to be as close as possible to the continuous case, we can use again the formula (I.2.5) on the last term, we obtain

$$\nabla\sigma\partial_t L(\star)\ \zeta + \nabla\left[\partial_v L(\star)\cdot\xi^\sigma\right] + \nabla\left[\zeta^\sigma\left(L(\star) - \partial_v L(\star)\cdot\Delta x\right)\right] - \zeta\nabla\left[L(\star) - \partial_v L(\star)\cdot\Delta x\right] = 0. \tag{V.3.11}$$

Taking the ∇-antiderivative of this expression, we deduce the conservation law (V.2.7). This concludes the proof. □

V.3.2 The σ-shifted case

The shifted case follows essentially the same line as the non shifted case. However, due to the shift, after the initial change of variables, one needs another rewriting of the invariance condition in order to identify the corresponding extended Lagrangian.

V.3.2.1 Rewriting the invariance condition and the extended Lagrangian

Following Section V.2.3, we have:

Lemma V.6. *Let the functional $\mathcal{L}^\sigma_{\Delta,\mathbb{T}}$ satisfying condition (VI.2), then we have*

$$\int_{t_a}^{t_b} L\left(t, x^\sigma(t), \Delta x(t)\right)\Delta t = \int_{t_a}^{t_b} L\left(g_s^0(t), \left[g_s^1\circ x\right]^\sigma(t), \Delta\left[g_s^1\circ x\right](t)\cdot\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t. \tag{V.3.12}$$

The proof is given in Section V.8.

However, in order to consider the time as a new variable in a shifted Lagrangian, one must rewrite the left-hand side of the invariance condition taking into account that ([8, Theorem 4, p.1224])

$$g_s^0(t) = (g_s^0)^\sigma(t) - \mu(t)\Delta g_s^0(t). \tag{V.3.13}$$

One then obtain:

Lemma V.7. *The invariance condition (V.3.12) can be written as*

$$\int_{t_a}^{t_b} L\left(t, x^\sigma(t), \Delta x(t)\right)\Delta t =$$
$$\int_{t_a}^{t_b} L\left((g_s^0)^\sigma(t) - \mu(t)\Delta g_s^0(t), \left[g_s^1\circ x\right]^\sigma(t), \Delta\left[g_s^1\circ x\right](t)\frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t \tag{V.3.14}$$

We then recover the computations made in ([8, p.1224], the proof of Theorem 4). We are now ready to introduce the extended Lagrangian.

V.3.2.2 Invariance of the extended Lagrangian

We define the **shifted extended Lagrangian** denoted by $\mathbb{L} : \mathbb{R} \times [a,b] \times \mathbb{R}^d \times \mathbb{R}^* \times \mathbb{R}^d \longrightarrow \mathbb{R}$ as

$$\mathbb{L}_\sigma(\tau; t, x, w, v) = L\left(t - \mu(\tau)w, x, \frac{v}{w}\right) w. \tag{V.3.15}$$

This shifted extended Lagrangian agree with the definition of the Lagrangian denoted $\tilde{L}$ in ([8, p.1224], the proof of Theorem 4).

We define the functional denoted by $\mathcal{L}_{\mathbb{L}_\sigma}$ as

$$\mathcal{L}_{\mathbb{L}_\sigma}(t,x) = \int_{t_a}^{t_b} \mathbb{L}_\sigma\left(\tau; t^\sigma(\tau), (x^\sigma \circ t)(\tau)), \Delta t(\tau), \Delta x(\tau)\right) \Delta\tau. \tag{V.3.16}$$

Taking into account the bundle path $\mathcal{F}$ defined in (V.3.4), we obtain that $\Delta[t] = 1$, so that the restriction of $\mathbb{L}_\sigma$ to $\mathcal{F}$ satisfies

$$\mathbb{L}_\sigma\left(\tau; t^\sigma(\tau) = \tau^\sigma, x^\sigma(\tau), \Delta\tau, \Delta x(\tau)\right) = L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right). \tag{V.3.17}$$

As a consequence, one can rewrite the invariance condition (V.3.14) as follows

Lemma V.8. *The invariance condition* (V.3.14) *over* $\mathcal{F}$ *can be written as*

$$\mathcal{L}_{\mathbb{L}_\sigma}(t,x) = \int_{t_a}^{t_b} \mathbb{L}_\sigma\left(\tau; \left[g_s^0\right]^\sigma(t(\tau)), \left[g_s^1 \circ x\right]^\sigma(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s} g_s^0(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1 \circ x\right](t(\tau))\right) \Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{V.3.18}$$

In [8, p.1225] the authors deduce from the previous result that they can apply the Noether theorem without transforming time (see [8, Theorem 3]) directly over the bundle paths $\mathcal{F}$. However, as in the previous nonshifted case, one must check that the solution of the Euler-Lagrange equation n ($\mathrm{EL}^{\Delta\circ\Delta}$) produce solution of the extended Lagrangian systems. As already mentioned in the introduction, this property is actually always satisfied in the continuous case due to specific properties of the differential calculus. In the time scale setting, this property fails.

Lemma V.9. *A path $\gamma = (t,x) \in \mathcal{F}$ is a critical point of $\mathcal{L}_{\mathbb{L}_\sigma}$ if, and only if, x is a critical point of $\mathcal{L}^\sigma_{\Delta,\mathbb{T}}$ and*

$$\Delta\left[\mathscr{H}(\cdot, x^\sigma, \Delta x\right](t) + \frac{\partial L}{\partial t}(t, x^\sigma, \Delta x) = 0,$$

where $H(t,x,v) = L(t,x,v) - v \cdot \partial_v L(t,x,v) - \mu(t)\partial_t L(t,x,v)$ is satisfied over the solutions of the shifted Euler-Lagrange equation for all $t \in \mathbb{T}^{\kappa^2}$.

The proof is given in Section V.8.

We deduce that a solution of the Euler-Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$) for L is a solution of the Euler-Lagrange equation for the extended Lagrangian $\mathbb{L}_\sigma$ over $\mathcal{F}$ if and only if it satisfies what is called the second Euler-Lagrange equation on time scales given in [7, Theorem 5, p.3]. This equation is implicitly assumed in [8] where the relation between the solutions of the Euler-Lagrange equation for L and those for $\mathbb{L}_\sigma$ over $\mathcal{F}$ is not discussed.

As a consequence, without any additional conditions on the solutions of n ($\mathrm{EL}^{\Delta\circ\Delta}$) for L one must reduce our attention not on the direct application of the Noether theorem without transforming time but to the infinitesimal invariance criterion formulated in [8, Theorem 2, p.1223]. This point is where our proof differs from the one given in [8].

V.3.2.3 Proof of the shifted time scales Noether Theorem

The infinitesimal invariance criterion ([8, Theorem 2, p.1223]) of the functional $\mathcal{L}_{\mathbb{L}_\sigma}$ over $\mathcal{F}$ is obtained by differentiating both sides of (V.3.18) around $s = 0$. We obtain

$$\partial_t \mathbb{L}_\sigma(\bullet)\zeta^\sigma(\tau) + \partial_x \mathbb{L}_\sigma(\bullet) \cdot \xi^\sigma(x) + \partial_w \mathbb{L}_\sigma(\bullet)\Delta\zeta(\tau) + \partial_v \mathbb{L}_\sigma(\bullet) \cdot \Delta\xi(x) = 0 \tag{V.3.19}$$

where $(\bullet) := \big(\tau; \tau^\sigma, x^\sigma(\tau), \Delta_{\widetilde{\mathbb{T}}}\tau, \Delta_{\widetilde{\mathbb{T}}}x(\tau)\big)$.

Substituting (V.8.12) into (V.3.19) gives

$$\begin{aligned}\partial_t L\,(\star_\sigma)\,\zeta^\sigma(\tau) &+ \partial_x L\,(\star_\sigma) \cdot \xi^\sigma(x) \\ &+ \Big[L\,(\star_\sigma) - \partial_v L\,(\star_\sigma)\,\Delta x(\tau) - \partial_t L\,(\star_\sigma)\,\mu(\tau)\Big]\Delta\zeta(\tau) + \partial_v L\,(\star_\sigma) \cdot \Delta\xi(x) = 0.\end{aligned} \tag{V.3.20}$$

Using the Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$) and the *time scales Leibniz rule*, we obtain

$$\begin{aligned}\partial_t L\,(\star_\sigma)\,\zeta^\sigma(\tau) + \big[L\,(\star_\sigma) - \partial_v L\,(\star_\sigma)\,\Delta x(\tau) - \partial_t L\,(\star_\sigma)\,\mu(\tau)\big]\Delta\zeta(\tau)& \\ + \Delta\big[\partial_v L\,(\star_\sigma) \cdot \xi(x)\big] = 0.&\end{aligned} \tag{V.3.21}$$

Observe that the term between brackets in (V.3.21) is the function $-\mathscr{H}$ defined in (V.2.19). Using the *time scales Leibniz rule*, we obtain

$$\big[-\mathscr{H}(\star_\sigma)\big]\Delta\zeta(\tau) = \Delta\big[-\mathscr{H}(\star_\sigma)\zeta(\tau)\big] + \Delta\big[\mathscr{H}(\star_\sigma)\big]\zeta^\sigma(\tau). \tag{V.3.22}$$

Substituting the formula (V.3.22) into (V.3.21) gives

$$\Big(\partial_t L\,(\star_\sigma) + \Delta\big[\mathscr{H}(\star_\sigma)\big]\Big)\zeta^\sigma(\tau) + \Delta\big[-\mathscr{H}(\star_\sigma)\zeta(\tau) + \partial_v L\,(\star_\sigma) \cdot \xi(x)\big] = 0. \tag{V.3.23}$$

We complete the proof by taking the Δ-antiderivative of this latter equation. □

V.4 Direct proof of the main results

We follow in this Section the usual proof of the Noether theorem consisting in deriving the invariance condition with respect to the parameter of the symmetry group and deducing a constant of motion.

V.4.1 The nonshifted case

Since the invariance condition (V.3.1) holds for any subinterval of $[a, b]$ and $x \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$, then we have:

$$L\,(t, x(t), \Delta x(t)) = L_s\left(g^0_s(t), (g^1_s \circ x)(t), \Delta\left(g^1_s \circ x\right)(t)\frac{1}{\Delta g^0_s(t)}\right)\Delta g^0_s(t).$$

Differentiating both sides of the latter equation with respect to s, it gives for $s = 0$ that

$$\zeta\partial_t L + \xi \cdot \partial_x L + (\Delta\xi - \Delta\zeta\Delta x) \cdot \partial_v L + \Delta\zeta\ L = 0. \tag{V.4.1}$$

Since this equation and the equation (V.3.8) are the same, one can follow the proof in subsection V.3.1.3.

Remark V.4 (Prolongation of vector fields in a time-scales setting)**.** *The operator appearing in* (V.4.1) *can be rewritten using the vector field denoted by* $\boldsymbol{X}^{(1)}$ *and defined by*

$$\boldsymbol{X}^{(1)} = \zeta\partial_t + \xi \cdot \partial_x + (\Delta\xi - \Delta\zeta\ \Delta x) \cdot \partial_v \tag{V.4.2}$$

By analogy with the definition of the prolongation of vector fields given by P. J. Olver (see [83, Definition 2.28, p.101]), we call this vector field the first prolongation of the vector field $\mathbf{X} = \zeta\partial_t + \xi\partial_x$. *Consequently, one can replace the condition* (V.3.1) *by the following invariance criterion*

$$\boldsymbol{X}^{(1)}L + \Delta\zeta\ L = 0. \tag{V.4.3}$$

In the case when $\mathbb{T} = \mathbb{R}$*, one recover the usual formula for the first prolongation (see [83, Theorem 2.36, p.110]) of the vector field* X*, i.e.,*

$$\boldsymbol{X}^{(1)} = \zeta\partial_t + \xi \cdot \partial_x + (\dot{\xi} - \dot{\zeta}\dot{x}) \cdot \partial_v. \tag{V.4.4}$$

In order to develop a full analogue of the theory of symmetries as presented in the book of P.J. Olver [83], one needs first to defined correctly the discrete analogue of vector fields which is still missing at that time.

V.4.2 The σ-shifted case

Since the invariance condition (V.3.12) holds for any subinterval of $[a, b]$ and $x \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$, then we have:

$$L\left(t, x^\sigma(t), \Delta x(t)\right) = L_s\left((g^0_s)^\sigma(t) - \mu(t)\Delta g^0_s(t), \left[g^1_s \circ x\right]^\sigma(t), \Delta\left[g^1_s \circ x\right](t)\frac{1}{\Delta g^0_s(t)}\right)\Delta g^0_s(t)$$

In the same way as done in the nonshifted case, by differentiating both sides of the above equation with respect to s, it gives for $s = 0$ that

$$\begin{aligned} 0 &= (\zeta^\sigma - \mu(t)\Delta\zeta)\partial_t L + \xi^\sigma \cdot \partial_x L + (\Delta\xi - \Delta x\Delta\zeta) \cdot \partial_v L + \Delta\zeta\ L \\ &= \zeta\partial_t L + \xi^\sigma \cdot \partial_x L + (\Delta\xi - \Delta x\Delta\zeta) \cdot \partial_v L + \Delta\zeta\ L. \end{aligned}$$

Since the latter equation and the equation (V.3.20) are the same, one can follow the same proof as in subsection V.3.2.3.

Remark V.5. *One can replace the condition* (V.3.12) *by an alternative condition that is*

$$\zeta\partial_t L + \xi^\sigma \cdot \partial_x L + (\Delta\xi - \Delta x\Delta\zeta) \cdot \partial_v L + \Delta\zeta\ L = 0. \tag{V.4.5}$$

V.5 Examples and simulations

V.5.1 The σ-shifted and nonshifted version of the Bartosiewicz and Torres example

We consider the Lagrangian introduced in [8] and given by

$$L(t, x, v) = \frac{x^2}{t} + tv^2, \quad \text{for} \quad x, v \in \mathbb{R}. \tag{V.5.1}$$

We discuss both the shifted and nonshifted Lagrangian functional associated to L and the corresponding conservation laws as obtained using the Noether theorem on time scales proved in the previous Section.

One can prove that the nonshifted Lagrangian functional possesses a variational symmetry given by:

Lemma V.10. *The Lagrangian functional associated to (V.5.1) is invariant under the family of transformation $G = \{g_s(t,x) = (te^s, x)\}_{s\in\mathbb{R}}$ where its infinitesimals are given by*

$$\zeta(t) = t \quad \textit{and} \quad \xi(x) = 0. \tag{V.5.2}$$

Proof: Indeed, we have $L\left(te^s, x, \frac{\Delta x}{e^s}\right)e^s = \left(\frac{x^2}{te^s} + te^s\frac{(\Delta x)^2}{e^{2s}}\right)e^s = L(t,x,\Delta x)$ so that condition (V.3.1) is satisfied. □

The same result is valid in the shifted case.

In the following, we consider two time scales given by

$$\mathbb{T}_1 = \{a+nh,\ n\in\mathbb{N}\},\ h=(b-a)/N, N\in\mathbb{N}^* \quad \text{and} \quad \mathbb{T}_2 = \{2^n,\ n\in\mathbb{N}\cup\{0\}\}, \tag{V.5.3}$$

which will be used to make simulations.

V.5.1.1 The nonshifted case

In our case, the (non-shifted) Euler–Lagrange equation associated with L is given by

$$\nabla\left[t\Delta x(t)\right] = \nabla\sigma(t)\frac{x}{t}, \tag{V.5.4}$$

with $\nabla\sigma(t) = 1$ if $t\in\mathbb{T}_1$ and $\nabla\sigma(t) = 2$ if $t\in\mathbb{T}_2$ and our time scales Noether's theorem generates the following first integral

$$I(t,x,v) = \sigma(t)\left(\frac{x^2}{t} - tv^2\right) + \int_a^t \left[-\nabla\sigma(t)\left(\frac{x^2}{t} - tv^2\right) - t\nabla\left(\frac{x^2}{t} - tv^2\right)\right]\nabla t. \tag{V.5.5}$$

V.5.1.2 The shifted case

We consider the following shifted Lagrangian

$$L\left(t, x^\sigma, v\right) = \frac{(x^\sigma)^2}{t} + tv^2 \tag{V.5.6}$$

and the family of transformation $G = \{\phi_s(t,x) = (te^s, x)\}_{s\in\mathbb{R}}$ which is a variational symmetry of L. Indeed, using the invariance criterion (V.4.5) we have that

$$t\left[-\left(\frac{x^\sigma}{t}\right)^2 + v^2\right] - 2tv^2 + \frac{(x^\sigma)^2}{t} + tv^2 = 0.$$

The (shifted) Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\Delta}$) associated to L is given by

$$\Delta\left[t\Delta x(t)\right] = \frac{x^\sigma}{t}. \tag{V.5.7}$$

According to the Noether theorem, we conclude that the following quantity

$$I(t,x^\sigma,v) = \sigma(t)\left(\frac{(x^\sigma)^2}{t} - tv^2\right) + \int_a^t \sigma(t)\left(-\frac{(x^\sigma)^2}{t^2} + v^2 + \Delta\left[\sigma(t)\left(-\frac{(x^\sigma)^2}{t^2} + v^2\right)\right]\right)\Delta t,$$

is a first integral.

Remark V.6. *In [8], the authors consider* $\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}$*. In that case,* $\sigma(t) = 2t$ *for all* $t \in \mathbb{T}$*, which gives the expression of* $C(t, x^\sigma, v)$ *in [8, Example 3], that is*

$$C(t, x^\sigma, v) = \sigma(t) \left(\frac{(x^\sigma)^2}{t} - tv^2 \right). \tag{V.5.8}$$

Numerical tests. With the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ as given before, we present simulations of both the Euler–Lagrange equations (V.5.4) and (V.5.7) which are called "approximate" on the picture as well as computations of the quantities $I(t, x, \Delta x)$ and $I(t, x^\sigma, \Delta x)$ on $\mathbb{T}_1$ and $\mathbb{T}_2$. In order to check the validity of our numerical scheme, we give also the exact solution of the Euler-Lagrange equation in the continuous case for the corresponding initial conditions.

As we can see in Figures V.4 and V.6 over the time scales $\mathbb{T}_1$ when h is sufficiently small, the solution of the nonshifted or shifted Euler-Lagrange equation provide very good approximations of the exact solution.

We can not expect such a result for the time scales $\mathbb{T}_2$ as in this case, the time increment is very big at the beginning of the simulation.

As expected, all the computations given in Figures V.4 and V.6 over $\mathbb{T}_1$ and V.5, and V.7 over $\mathbb{T}_2$ show that the quantities obtained in the Noether theorem on time scales are constant over the solutions of the time scales Euler–Lagrange equation (V.5.4) and (V.5.7) respectively.

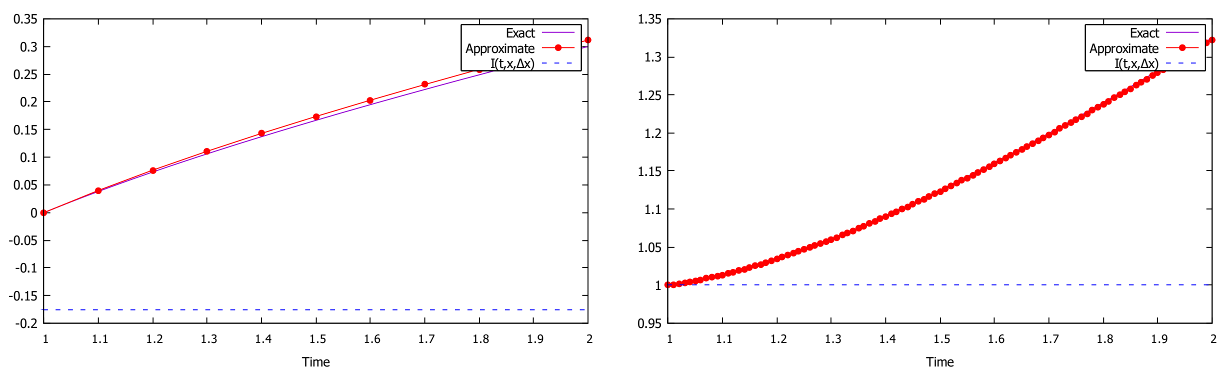


Figure V.4: Numerical solution of (V.5.4) and the quantity (V.5.5) on time scales $\mathbb{T}_1$. (left) $x_0 = 0, \Delta x_0 = 0.4, h = 0.1$. (right) $x_0 = 1, \Delta x_0 = 0.1, h = 0.01$.

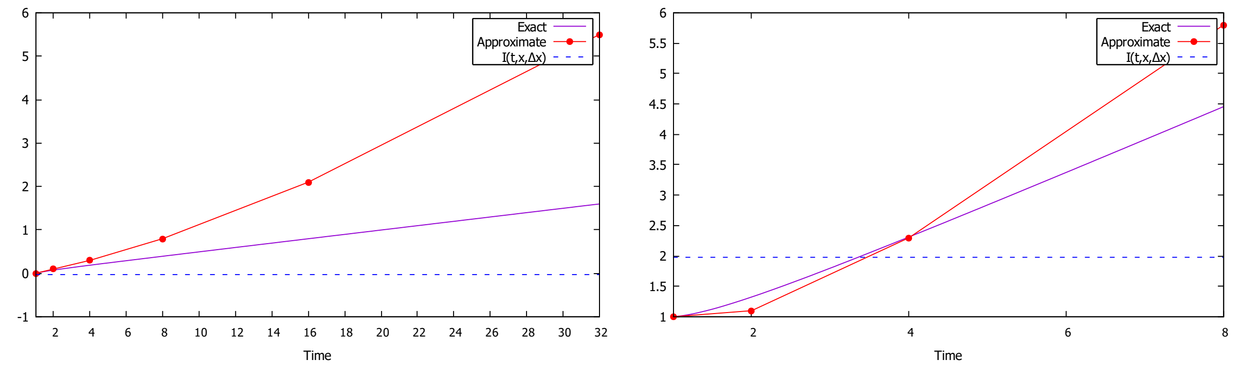


Figure V.5: Numerical solution of (V.5.4) and the quantity (V.5.5) on time scales $\mathbb{T}_2$. (left) $n = 5, x_0 = 0, \Delta x_0 = 0.1$. (right) $n = 3, x_0 = 1, \Delta x_0 = 0.1$.

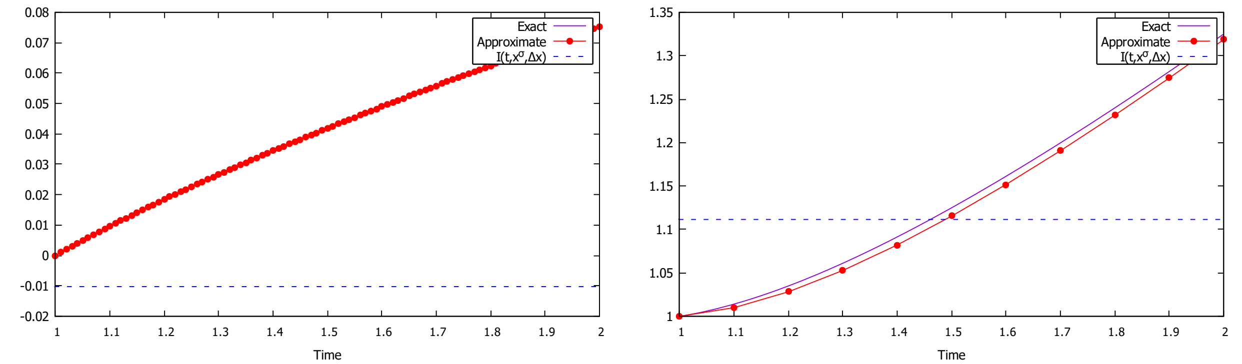


Figure V.6: Numerical solution of (V.5.7) and the quantity $I(t, x^\sigma, v)$ on $\mathbb{T}_1$. (left) $x_0 = 0, \Delta x_0 = 0.1, h = 0.01$. (right) $x_0 = 1, \Delta x_0 = 0.1, h = 0.1$.

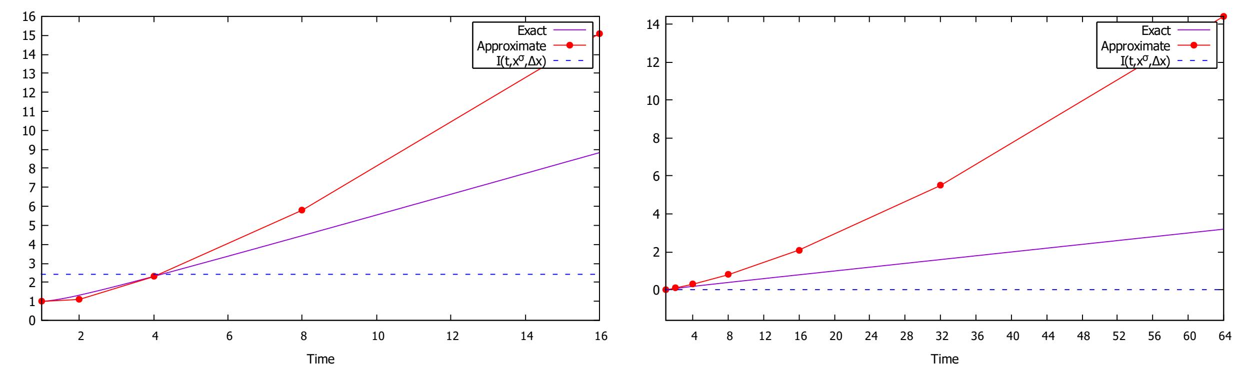
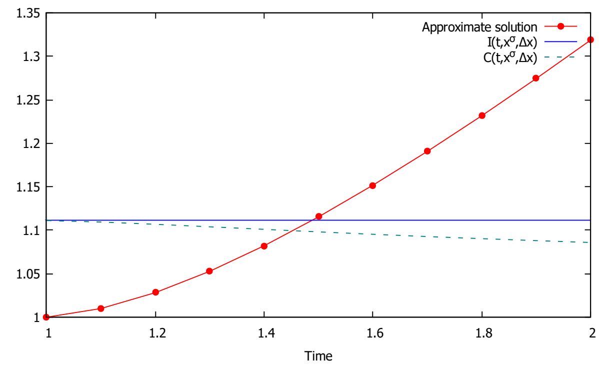


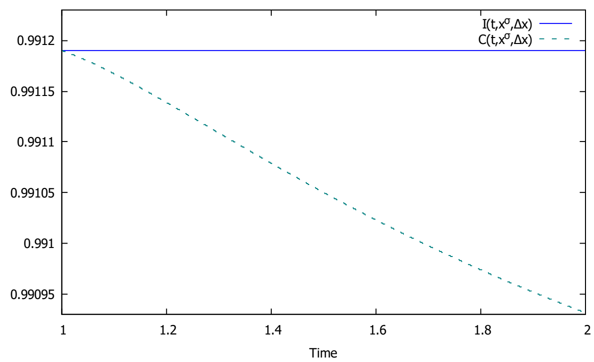
Figure V.7: Numerical solution of (V.5.7) and the quantity $I(t, x^\sigma, v)$ on $\mathbb{T}_2$. (left) $x_0 = 1, \Delta x_0 = 0.1, n = 4$. (right) $x_0 = 0, \Delta x_0 = 0.1, n = 6$

V.5.1.3 Comparison between Bartosiewicz & Torres result and our result

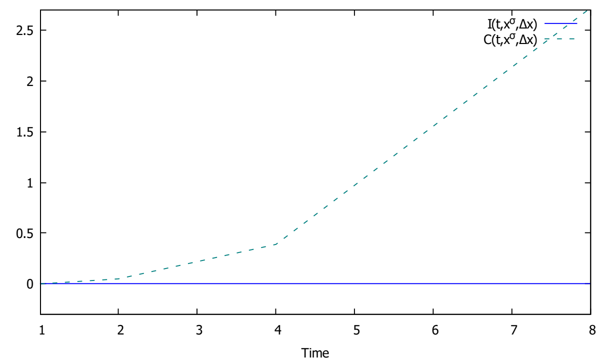
As we have seen, the quantity $I(t, x^\sigma, \Delta x)$ is a constant of motion over the solution of the time scales Euler–Lagrange equation (V.5.7). It is clearly not the case for the quantity $C(t, x^\sigma, \Delta x)$ provided by the Noether theorem in [8].



(a) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.1$



(b) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.001$

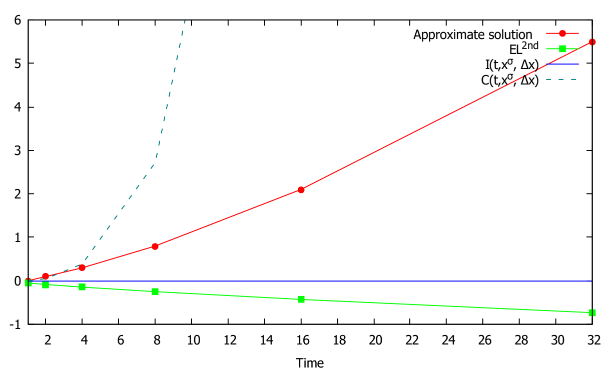


(c) On $\mathbb{T}_2$, $x_0 = 0, \Delta x_0 = 0.1, n = 3$

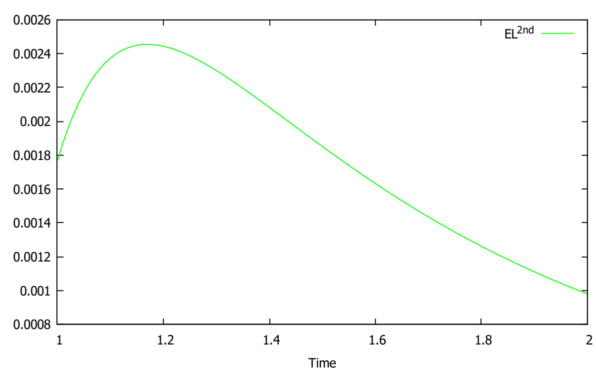
Figure V.8: The trace of $I(t, x^\sigma, v)$ and $C(t, x^\sigma, v)$ on time scales $\mathbb{T}_1$ and $\mathbb{T}_2$

V.5.1.4 Numerical test of the second Euler–Lagrange equation

In [8], the authors require for the quantity $C(t, x^\sigma, v)$ to be a constant of motion over the solution of (V.5.7) that the second Euler–Lagrange equation must be satisfied. We then test the equality to zero of the left-hand side of the equation. We obtain the following green lines for the time-scales $\mathbb{T}_1$ and $\mathbb{T}_2$



(a) On $\mathbb{T}_2$, $x_0 = 0, \Delta x_0 = 0.1, n = 5$



(b) On $\mathbb{T}_1$, $x_0 = 1, \Delta x_0 = 0.1, h = 0.01$

Figure V.9: Behavior of the second Euler–Lagrange, $I(t, x^\sigma, v)$ and $C(t, x^\sigma, v)$.

proving that the second Euler-Lagrange equation is not satisfied.

V.5.2 The Kepler problem in the plane and a result of X.H. Zhai and L.Y. Zhang

We consider the time scales analogue of the *Kepler problem* in the plane already studied by X.H. Zhai and L.Y. Zhang in ([90], Example 1).

We consider the Lagrangian defined on $\big(\mathbb{R}^2 \setminus \{(0,0)\}\big) \times \mathbb{R}^2$ by

$$L(x_1, x_2, v_1, v_2) = \frac{1}{2}(v_1^2 + v_2^2) + \frac{1}{\sqrt{x_1^2 + x_2^2}}, \tag{V.5.9}$$

which corresponds to the Lagrangian of the Kepler problem of two interacting particle with one of mass one under the gravitational field in the plane where one of the particle is positioned at the origin.

A time scales analogue of the Kepler problem in the shifted calculus of variation setting is then associated to the functional

$$\mathcal{L}_{\Delta,\mathbb{T}}(x) = \int_a^b \left[\frac{1}{2}(\Delta[x_1])^2 + (\Delta[x_2])^2) + \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}}\right] \Delta t. \tag{V.5.10}$$

The Euler–Lagrange equations are given by

$$\begin{cases} \Delta \circ \Delta[x_1] &= -\dfrac{x_1^\sigma}{((x_1^\sigma)^2 + (x_2^\sigma)^2)^{3/2}}, \\ \Delta \circ \Delta[x_2] &= -\dfrac{x_2^\sigma}{((x_1^\sigma)^2 + (x_2^\sigma)^2)^{3/2}}. \end{cases} \tag{V.5.11}$$

Moreover the Hamiltonian function associated to (V.5.11) is given by

$$H(x_1, x_2, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}}. \tag{V.5.12}$$

One easily shows that the group of rotations

$$g_s(x_1, x_2) = \left(x_1 \cos(s) - x_2 \sin(s), x_1 \sin(s) + x_2 \cos(s)\right), \tag{V.5.13}$$

for $s \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{R}^2$ is a variational symmetry of the functional on any time scales $\mathbb{T}$. Indeed, we have for all $s \in \mathbb{R}$, $x = (x_1, x_2) \in \mathrm{C}^{1,\Delta}_{\mathrm{rd}}(\mathbb{T})$ and $t \in \mathbb{T}^\kappa$

$$L(x, \Delta x) = L(g_s(x), \Delta\left[g_s(x)\right]), \tag{V.5.14}$$

as $\Delta\left[g_s(x)\right] = g_s\left(\Delta[x]\right)$ by linearity and continuity of g_s with respect to x, and the fact that g_s is an isometry. The invariance of the functional then follows.

As $\frac{\partial g_s}{\partial s}(x_1, x_2)|_{s=0} = (-x_2, x_1)$, the Noether theorem on time scales then ensure that the function

$$I_1(\cdot, x(\cdot)) = -x_2\, \Delta[x_1] + x_1\, \Delta[x_2], \tag{V.5.15}$$

is a first integral of the time scales equation (V.5.11). This result coincide with the one given by X.H. Zhai and L.Y. Zhang in ([90], equation (45)).

It is clear that the group of time translations is a variational symmetry of (V.5.10), since this functional does not depend on the time. Then, our Noether theorem on time scales produces the following first integral

$$\begin{aligned} I_2(\cdot, x(\cdot)) &= -H(x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2) + \int_a^t \Delta H(x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2) \Delta t \\ &= -H(x_1^\sigma(a), x_2^\sigma(a), \Delta x_1(a), \Delta x_2(a)) \end{aligned} \tag{V.5.16}$$

Indeed, if we consider the uniform time scales $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$ on the interval $[0; 3.5]$ with $h = 0.1$ and the initial conditions are $x_1 = 1, x_2 = 0, v_1 = v_2 = 1$, we obtain the following simulation

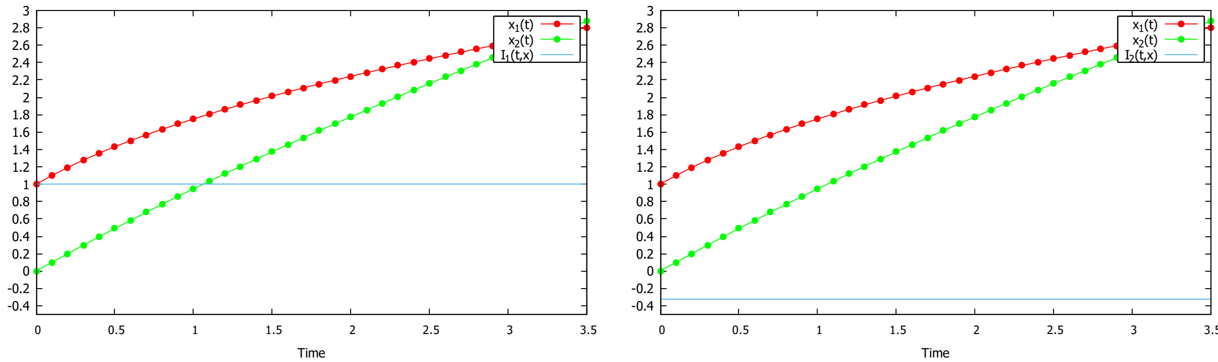
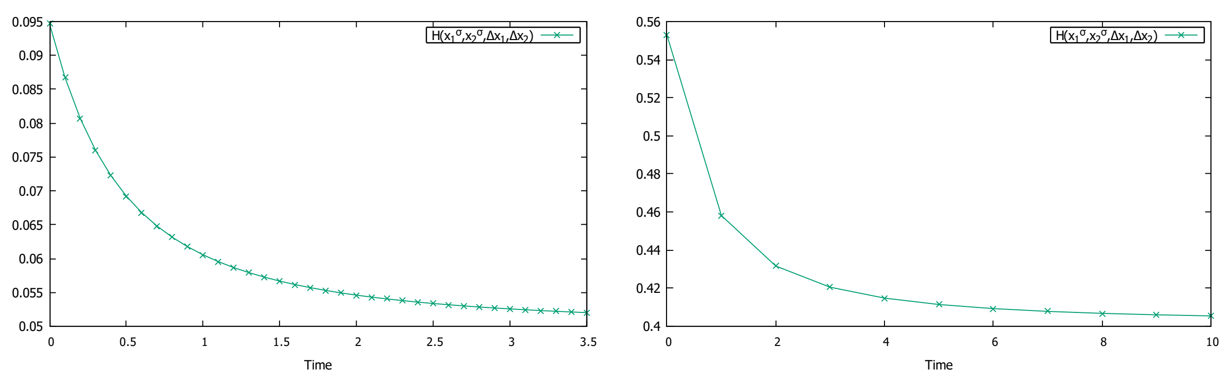


Figure V.10: Simulation of the quantities $I_1(t, x)$ and $I_2(t, x)$.

However, as for the Z. Bartosiewicsz and D.F.M. Torres example [8], a problem occurs with time dependent group of transformations. Namely, X.H. Zhai and L.Y. Zhang asserts that the Hamiltonian is a constant of motion on the solutions of (V.5.11), i.e., that the quantity

$$H(t, x_1^\sigma, x_2^\sigma, \Delta x_1, \Delta x_2) = \frac{1}{2}\left((\Delta[x_1])^2 + (\Delta[x_2])^2\right) - \frac{1}{\sqrt{(x_1^\sigma)^2 + (x_2^\sigma)^2}}, \tag{V.5.17}$$

is a constant on the solution of the equation for an arbitrary time scale. This is of course the case for any continuous time scales $\mathbb{T} = [a, b]$ but not the case for other time scales like $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$. Indeed, in this case, one obtain the following simulation



(a) Uniform time scales on $[0; 3.5]$ with $h = 0.1$ **(b)** Uniform time scales on $[0, 10]$ with $h = 1$

Figure V.11: Simulation of the Hamiltonian function (V.5.17) on a uniform time scales over the solutions (V.5.11).

V.6 Caputo duality principle and a time scales Noether's Theorem for the nabla calculus of variations

In this section, some properties, basic definitions about *Caputo's duality principle* are presented and such principle was also applied to the calculus of variations on time scales.

We refer to [19] which contain more details and proofs on *Caputo's duality principle.*

V.6.1 Reminder about Caputo duality principle

Definition V.4. *Let $\mathbb{T}$ be a time scales. The dual time scales of $\mathbb{T}$ is a new time scales defined by* $\mathbb{T}^* := \{\tau \in \mathbb{R} : -\tau \in \mathbb{T}\}$.

Definition V.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function defined on a time scales $\mathbb{T}$. The dual function $f^* : \mathbb{T}^* \to \mathbb{R}$ is defined by $f^*(\tau) = f(-\tau)$ for all $\tau \in \mathbb{T}^*$.*

Let $\mathbb{T}$ be a time scales. If $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ denote, respectively, the forward and backward jump operators on $\mathbb{T}$, then we denote to the forward and backward jump operators on $\mathbb{T}^*$, respectively, by $\hat{\sigma}, \hat{\rho} : \mathbb{T}^* \to \mathbb{T}^*$.

Let μ (resp. ν) the forward (resp. the backward) graininess on $\mathbb{T}$, we denote by $\hat{\mu}$ (resp. $\hat{\nu}$), the forward (resp. the backward) graininess on $\mathbb{T}^*$.

Let Δ (resp. ∇) be the delta (resp. the nabla) derivative on $\mathbb{T}$, we denote by $\widehat{\Delta}$ (resp. $\widehat{\nabla}$) the delta (resp. the nabla) derivative on $\mathbb{T}^*$.

Proposition V.2. *Let $\mathbb{T}$ be a time scales with $a, b \in \mathbb{T}$, $a < b$ and let $f : \mathbb{T} \to \mathbb{R}$ a function. We have the following:*

- $(\mathbb{T}^\kappa)^* = (\mathbb{T}^*)_\kappa$ *and* $(\mathbb{T}_\kappa)^* = (\mathbb{T}^*)^\kappa$
- $([a,b])^* = [-b,-a]$ *and* $([a,b]^\kappa)^* = [-b,-a]_\kappa \subseteq \mathbb{T}^*$.
- *For all $\tau \in \mathbb{T}^*$, $\hat{\sigma}(\tau) = -\rho(-\tau) = -\rho^*(\tau)$ and $\hat{\rho}(\tau) = -\sigma(-\tau) = -\sigma^*(\tau)$.*
- *For all $\tau \in \mathbb{T}^*$, $\hat{\mu}(\tau) = \nu^*(\tau)$ and $\hat{\nu}(\tau) = \mu^*(\tau)$.*
- *Given a function $f : \mathbb{T} \to \mathbb{R}$ and its dual $f^* : \mathbb{T}^* \to \mathbb{R}$. Then, $f \in \mathrm{C}^0_{\mathrm{rd}}(\mathbb{T})$ (resp. $f \in \mathrm{C}^0_{\mathrm{ld}}(\mathbb{T})$) if and only if $f^* \in \mathrm{C}^0_{\mathrm{ld}}(\mathbb{T}^*)$ (resp. $f^* \in \mathrm{C}^0_{\mathrm{rd}}(\mathbb{T}^*)$).*
- *If f is Δ (resp. ∇) differentiable at $t \in \mathbb{T}^\kappa$ (resp. at $t \in \mathbb{T}_\kappa$), then $f^* : \mathbb{T}^* \to \mathbb{R}$ is ∇ (resp. Δ) differentiable at $-t \in (\mathbb{T}^*)_\kappa$ (resp. $-t \in (\mathbb{T}^*)^\kappa$), and*

$$\begin{aligned}
\Delta f(t) &= -\widehat{\nabla} f^*(-t), \quad (\textit{resp.} \nabla f(t) = -\widehat{\Delta} f^*(-t)),\\
\Delta f(t) &= -\left(\widehat{\nabla} f^*\right)^*(t), \quad (\textit{resp.} \nabla f(t) = -\left(\widehat{\Delta} f^*\right)^*(t)),\\
(\Delta f)^*(-t) &= -\widehat{\nabla} f^*(-t), \quad (\textit{resp.} (\nabla f)^*(-t) = -\widehat{\Delta} f^*(-t)).
\end{aligned}$$

- *If $f : [a,b] \to \mathbb{R}$ is rd-continuous, then*

$$\int_a^b f(t)\Delta t = \int_{-b}^{-a} f^*(\tau)\widehat{\nabla}\tau.$$

- *If $f : [a,b] \to \mathbb{R}$ is ld-continuous, then*

$$\int_a^b f(t)\nabla t = \int_{-b}^{-a} f^*(\tau)\widehat{\Delta}\tau.$$

Definition V.6. *Let $L : \mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be a Lagrangian. Then, the corresponding dual lagrangian $L^* : \mathbb{T}^* \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$L^*(\tau, x, v) = L(-\tau, x, -v) \quad \text{for all} \quad (\tau, x, v) \in \mathbb{T}^* \times \mathbb{R}^n \times \mathbb{R}^n.$$

One can notice that,

$$\partial_t L^*(\tau, x, v) = -\partial_t L(-\tau, x, -v), \tag{V.6.1}$$
$$\partial_x L^*(\tau, x, v) = \partial_x L(-\tau, x, -v), \tag{V.6.2}$$
$$\partial_v L^*(\tau, x, v) = -\partial_v(-\tau, x, -v). \tag{V.6.3}$$

V.6.2 A time scales Noether's theorem for the nabla nonshifted calculus of variations

Consider the functional $\mathcal{L}_{\nabla,\mathbb{T}} : \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}_{\nabla,\mathbb{T}}(x) = \int_a^b L\left(t, x(t), \nabla x(t)\right)\nabla t \tag{V.6.4}$$

where $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Lagrangian on the time scales $\mathbb{T}$.

Theorem V.3 (Euler–Lagrange equation - Nonshifted case [14])**.** *Assume that ρ is Δ-differentiable on $\mathbb{T}^\kappa$. Then, the critical points of the functional (V.6.4) are solutions of the following Euler–Lagrange equation*

$$\Delta\left[\frac{\partial L}{\partial v}(t, x(t), \nabla x(\tau))\right] = \Delta\rho(t)\frac{\partial L}{\partial x}(t, x(t), \nabla x(t)), \tag{EL$^{\Delta\circ\nabla}$}$$

for every $t \in \mathbb{T}^\kappa_\kappa$.

Theorem V.4 (Noether's Theorem - Nonshifted case)**.** *Let $\mathbb{T}$ be a time scales such that ρ is Δ-differentiable on $\mathbb{T}^\kappa$. Let $G = \{g_s(t,x) = (g^0_s(t), g^1_s(x))\}_{s\in\mathbb{R}}$ a $(\nabla,\mathbb{T})$-variational symmetry of the functional (V.6.4) with the corresponding infinitesimal generator given by*

$$\mathbf{X}\zeta(t)\frac{\partial}{\partial t} + \xi(x)\cdot\frac{\partial}{\partial x}. \tag{V.6.5}$$

Then, the function

$$\bar{I}(t,x,v) = -\zeta^\rho(t)\cdot H(\star) + \xi^\rho(x)\cdot\partial_v L(\star) + \int_a^t \zeta(t)\Big[\Delta\rho(t)\partial_t L(\star) + \Delta\big(H(\star)\big)\Big]\,\Delta t, \tag{V.6.6}$$

where H is defined in (V.2.8) and $(\star) = (t, x(t), \nabla x(t))$, is a constant of motion over the solution of the time scales Euler–Lagrange equation (EL$^{\Delta\circ\nabla}$), i.e., that

$$\Delta\big[I\left(\cdot, x(\cdot), \nabla x(\cdot)\right)\big](t) = 0, \tag{V.6.7}$$

for all solutions x of (EL$^{\Delta\circ\nabla}$) and any $t \in \mathbb{T}^\kappa_\kappa$. .

V.6.3 A time scales Noether's theorem for the nabla shifted calculus of variations

Consider the following functional $\mathcal{L}^{\rho}_{\nabla,\mathbb{T}}, \mathbb{T} : \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T}) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}^{\rho}_{\nabla,\mathbb{T}}(x) = \int_a^b L\left(t, x^{\rho}(t), \nabla x(t)\right) \nabla t \tag{V.6.8}$$

where $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Lagrangian on the time scales $\mathbb{T}$.

Theorem V.5 (Euler–Lagrange equation - ρ-shifted case)**.** *The critical points of $\mathcal{L}^{\rho}_{\nabla,\mathbb{T}}$ are solutions of the following Euler–Lagrange equation*

$$\nabla\left[\frac{\partial L}{\partial v}(\cdot, x^{\rho}, \nabla x)\right](t) = \frac{\partial L}{\partial x}(t, x^{\rho}(t), \nabla x(t)), \tag{EL$^{\nabla\circ\nabla}$}$$

for every $t \in \mathbb{T}_{\kappa}$.

Theorem V.6 (Noether's Theorem - ρ-shifted case)**.** *Let $\mathbb{T}$ be a time scales and let $G = \{g_s(t,x) = (g^0_s(t), g^1_s(x))\}_{s\in\mathbb{R}}$ a $(\nabla,\mathbb{T})$-admissible projectable group of transformations be a variational symmetry of $\mathcal{L}^{\rho}_{\nabla,\mathbb{T}}$ and let the corresponding infinitesimal generator given by*

$$\mathbf{X} = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\cdot\frac{\partial}{\partial x}. \tag{V.6.9}$$

Then, the function

$$\bar{I}(t, x^{\rho}, v) = -\zeta(t)\cdot\bar{\mathscr{H}}(\star_{\rho}) + \xi(x)\cdot\partial_v L(\star_{\rho}) + \int_a^t \zeta^{\rho}(t)\Big[\partial_t L(\star_{\rho}) + \nabla\big(\bar{\mathscr{H}}(\star_{\rho})\big)\Big]\nabla t, \tag{V.6.10}$$

where $\bar{\mathscr{H}} : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ is defined by $\bar{\mathscr{H}}(t,x,v) = H(t,x,v) - \partial_t L(t,x,v)\nu(t)$ and $(\star_{\rho}) = (t, x^{\rho}(t), \nabla x(t))$, is a constant of motion over the solution of the time scales Euler–Lagrange equation, i.e., that

$$\nabla\big[I\left(\cdot, x(\cdot), \nabla x(\cdot)\right)\big](t) = 0, \tag{V.6.11}$$

for all solutions x of (EL$^{\nabla\circ\nabla}$) and any $t \in \mathbb{T}_{\kappa}$.

V.6.4 Example and simulations

Consider the time scales $\mathbb{T} = \{t_k = a + kh,\ k \in \mathbb{N}\}$ and the following Lagrangian [56]

$$L(t,x,v) = L(t,x,v) = \frac{t}{2}\left(v^2 - 2e^x\right),$$

then the corresponding Euler–Lagrange equation is given by

$$\Delta(t\nabla x) = -te^x.$$

The family of transformation $G = \{g_s(t,x) = (te^s, x - 2s)\}_{s\in\mathbb{R}}$ where its infinitesimal generator is given by

$$\mathbf{X} = t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial x}$$

is a variational symmetry of L. Indeed, we have

$$L\left(e^s t, x - 2s, \frac{\nabla x}{e^s}\right)e^s = \frac{e^s t}{2}\left[\left(\frac{\nabla x}{e^s}\right)^2 - 2e^{x-2s}\right]e^s = L\left(t, x, \nabla x\right).$$

Therefore, Noether's theorem gives the following conservation law

$$\bar{I}(t,x,v) = -\rho(t)\ \frac{t}{2}\left(v^2+2e^x\right) - 2tv + \int_a^t \frac{t}{2}\left[\left(v^2-2e^x\right)+\Delta\left(t\left(v^2+2e^x\right)\right)\right]\Delta t. \quad \text{(V.6.12)}$$

In a shifted case, consider the following Lagrangian

$$L(t,x^\rho,v) = \frac{t}{2}\left(v^2 - 2e^{x^\rho}\right), \quad \text{(V.6.13)}$$

with the (shifted) Euler–Lagrange equation is given by

$$\nabla(\nabla x) = -te^{x^\rho}. \quad \text{(V.6.14)}$$

Using the invariance criterion of the functional $\mathcal{L}^\rho_{\nabla,\mathbb{T}}$ given by [75]

$$\zeta\frac{\partial L}{\partial t} + \xi^\rho\frac{\partial L}{\partial x} + (\nabla\xi - \nabla x\nabla\zeta)\frac{\partial L}{\partial v} + \nabla\zeta\ L = 0, \quad \text{(V.6.15)}$$

one check that the family of transformation $G = \{g_s(t,x) = (te^s, x-2s)\}_{s\in\mathbb{R}}$ is also a variational symmetry of (V.6.13). The Noether theorem gives the following conservation law:

$$\bar{I}(t,x^\rho,v) = -t\ \bar{\mathscr{H}}(t,x^\rho,v) - 2tv + \int_a^t \rho(t)\left[\frac{1}{2}\left(v^2-2e^x\right)+\nabla\bar{\mathscr{H}}(t,x^\rho,v)\right]\nabla t, \quad \text{(V.6.16)}$$

where $\bar{\mathscr{H}}(t,x^\rho,v) = \frac{\rho(t)}{2}\left(v^2-2e^{x^\rho}\right) + 2te^{x^\rho}$.

Simulations of the quantities $\bar{I}(t,x,v)$ and $\bar{I}(t,x^\rho,v)$ over $\mathbb{T}$ with $x_0 = 1, v_0 = 0.1$ and $h = 10$ give:

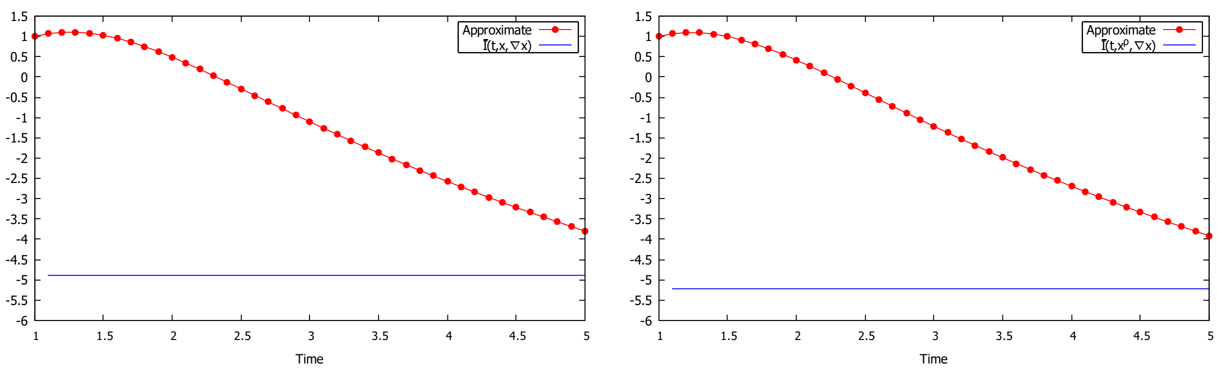


Figure V.12: The simulation of $\bar{I}(t,x,v)$ and $\bar{I}(t,x^\rho,v)$.

V.6.5 Comparison with the work of N. Martins and D.F.M. Torres

Applying the result of N. Martins and D.F.M. Torres in [75, Theorem 3.4] on our example, they assert that the quantity

$$M(t,x^\rho,v) = -t\ \frac{\rho(t)}{2}\left(v^2-2e^{x^\rho}\right) + 2te^{x^\rho} - 2tv \quad \text{(V.6.17)}$$

is constant of motion over the solutions of (V.6.14). The simulations then gives the following results:

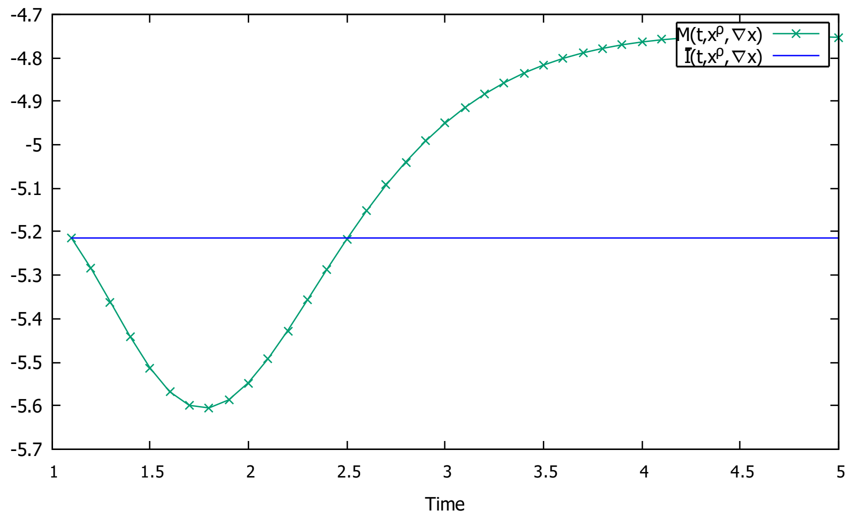


Figure V.13: $x_0 = 1, v = 0.1, h = 0.1$

We clearly see that M is not constant.

V.7 Proof of the main results using the Caputo duality principle

We give the proof for the nonshifted case and for the shifted one can be treated in the same manner.

V.7.1 The nonshifted case

Lemma V.11. *Let $L : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous Lagrangian. Then*

$$\int_a^b L\left(t, x(t), \nabla x(t)\right) \nabla t = \int_\alpha^\beta L^*\left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right) \widehat{\Delta}\tau, \tag{V.7.1}$$

for all function $x \in \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$, where $\alpha = -b$ and $\beta = -a$.

The proof of this lemma is immediate from the last property of Proposition V.2 and Definition V.6 and the point of this lemma in the following (see [75]). $x \in \mathrm{C}^{1,\nabla}_{\mathrm{ld}}(\mathbb{T})$ is a critical point of the functional (V.6.4) if and only if $x^* \in \mathrm{C}^{1,\hat{\Delta}}_{\mathrm{rd}}(\mathbb{T}^*)$ is a critical point of the functional

$$\mathcal{L}_{\widehat{\Delta},\mathbb{T}^*}(y) = \int_\alpha^\beta L^*\left(\tau, y(\tau), \widehat{\Delta} y(\tau)\right) \widehat{\Delta}\tau, \quad \text{with} \quad \alpha = -b,\ \beta = -a. \tag{V.7.2}$$

V.7.1.1 Proof of Euler–Lagrange equation ($\mathrm{EL}^{\Delta\circ\nabla}$)

The proof follows from the previous lemma. Let $\hat{\sigma}$ be $\widehat{\nabla}$-differentiable on $(\mathbb{T}^\kappa)^*$, Let $x^* \in \mathrm{C}^{1,\hat{\Delta}}_{\mathrm{rd}}(\mathbb{T}^*)$ be a critical point of the functional (V.7.2), then

$$\widehat{\nabla}\left[\frac{\partial L^*}{\partial v}\left(\cdot, x^*, \widehat{\Delta} x^*\right)\right](\tau) = \widehat{\nabla}\hat{\sigma}(\tau) \frac{\partial L^*}{\partial x}\left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right), \tag{V.7.3}$$

for all $\tau \in (\mathbb{T}^\kappa_\kappa)^*$.

According to the relations (V.6.1) and (V.6.3), we have that:

$$\partial_x L^* \left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right) = \partial_x L \left(-\tau, x(-\tau), \nabla x(-\tau)\right) \tag{V.7.4}$$

$$\partial_v L^* \left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right) = -\partial_v L \left(-\tau, x(-\tau), \nabla x(-\tau)\right). \tag{V.7.5}$$

Now, let us take $P(\tau) = \partial_v L^* \left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right)$ and $Q(\tau) = \partial_v L \left(\tau, x(\tau), \nabla x(\tau)\right)$, then the equation (V.7.5) can be written as

$$P(\tau) = -Q^*(\tau),$$

so that,

$$\widehat{\nabla} P(\tau) = -\widehat{\nabla} Q^*(\tau) = \Delta Q(-\tau).$$

Since $\tau \in (\mathbb{T}^\kappa_\kappa)^*$, then we get by taking $t = -\tau$ that $t \in \mathbb{T}^\kappa_\kappa$ and with the help of Proposition V.2 we deduce that ρ is Δ-differentiable at t. Finally, using the relation $\widehat{\nabla}\hat{\sigma}(\tau) = \Delta\rho(-\tau)$ and (V.7.3) we obtain

$$\Delta Q(t) = \Delta\rho(t) \frac{\partial L}{\partial x}(t, x(t), \nabla x(t)).$$

This complete the proof. □

V.7.1.2 Proof of Noether Theorem V.4

It follows from [75] that, if G is a variational symmetry of the functional (V.6.4) with the corresponding infinitesimal generator $\mathbf{X} = \zeta\partial_t + \xi\partial_x$, then the group G^* defined by

$$\begin{cases} (g^*)^0_s(\tau) = \tau - s\zeta^*(\tau), \\ (g^*)^1_s(x) = y + s\xi^*(y), \end{cases} \tag{V.7.6}$$

where, $\zeta^*(\tau) = \zeta(-\tau)$ and $\xi^*(y) = \xi(y)$ is a variational symmetry of the functional (V.7.2). Then, applying Theorem VI.4 to the functional (V.7.2), we have from (V.2.7) that the function

$$I^*(\tau, x^*) = (\zeta^*)^{\hat{\sigma}}(\tau) H^*[x^*](\tau) + \xi^{\hat{\sigma}}(x^*) \cdot \partial_v L^*[x^*](\tau) - \int_a^\tau \zeta^* \left[\widehat{\nabla}\hat{\sigma}\partial_t L^*[x^*](\tau) + \widehat{\nabla}\left(H^*[x^*](\tau)\right)\right] \widehat{\nabla}\tau, \tag{V.7.7}$$

is constant over the solution of (V.7.3), i.e.,

$$\widehat{\nabla}\left[(\zeta^*)^{\hat{\sigma}}(\tau) H^*[x^*](\tau) + \xi^{\hat{\sigma}}(x^*) \cdot \partial_v L^*[x^*](\tau)\right] - \zeta^*\widehat{\nabla}\hat{\sigma}\partial_t L^*[x^*](\tau) - \zeta^*\widehat{\nabla}\left(H^*[x^*](\tau)\right) = 0, \tag{V.7.8}$$

where $[x^*](\tau) = \left(\tau, x^*(\tau), \widehat{\Delta} x^*(\tau)\right)$ and $H^*[x^*](\tau) = L^*[x^*](\tau) - \partial_v L^*[x^*](\tau) \cdot \widehat{\Delta} x^*(\tau)$.
For simplicity, let $[x](\tau) = (\tau, x(\tau), \nabla x(\tau))$, $Q(\tau) = \partial_v L[x](\tau)$, $T(\tau) = \partial_t L[x](\tau)$ and $Z(\tau) = Q(\tau) \cdot \nabla x(\tau) - L[x](\tau)$. Taking in your mind the relations:

$$\begin{aligned} (\zeta^*)^{\hat{\sigma}}(\tau) &= (\zeta^\rho)^*(\tau), \quad \xi^{\hat{\sigma}}(x^*(\tau)) = (\xi^\rho \circ x)^*(\tau), \\ \widehat{\Delta} x^*(\tau) &= -\nabla x(-\tau), \quad \widehat{\nabla}\hat{\sigma}(\tau) = (\Delta\rho)^*(\tau) = \Delta\rho(-\tau) \\ \partial_v L^*[x^*](\tau) &= -\partial_v L\left(-\tau, x(-\tau), \nabla x(-\tau)\right) = -\partial_v L[x](-\tau) = -Q^*(\tau) \\ \partial_t L^*[x^*](\tau) &= -\partial_t L\left(-\tau, x(-\tau), \nabla x(-\tau)\right) = -\partial_t L[x](-\tau) = -T^*(\tau) \\ H^*[x^*](\tau) &= Q(-\tau) \cdot \nabla x(-\tau) - L[x](-\tau) = Z^*(\tau). \end{aligned}$$

we have the term $\widehat{\nabla}[\cdots]$ in (V.7.8) becomes

$$\widehat{\nabla}\Big((\zeta^*)^{\hat\sigma}(\tau)Z^*(\tau)+\xi^{\hat\sigma}(x^*)\cdot\partial_v L^*[x^*](\tau)\Big)=\widehat{\nabla}\left(\zeta^\rho\ Z\right)^*(\tau)-\widehat{\nabla}\Big(\left(\xi^\rho\circ x\right)^*\cdot Q^*\Big)(\tau)$$
$$=-\Delta\left(\zeta^\rho\cdot Z\right)(-\tau)+\Delta\left(\xi^\rho(x)\cdot Q\right)(-\tau),$$

and the rest terms, we have

$$\zeta^*(\tau)\widehat{\nabla}\hat\sigma(\tau)\partial_t L^*[x^*](\tau)=-\left(\zeta\ \Delta\rho\ T\right)^*(\tau),$$
$$\zeta^*(\tau)\widehat{\nabla}\left(H^*[x^*](\tau)\right)=-\left(\zeta\ \Delta Z\right)^*(\tau).$$

Substituting all of these formulas into (V.7.8) and replacing $-\tau$ by $t\in\mathbb{T}^\kappa_\kappa$ gives

$$\Delta\Big(-\zeta^\rho(t)\ Z(t)+\xi^\rho(x(t))\cdot Q(t)\Big)+\zeta(t)\Big(\Delta\rho(t)\ T(t)+\Delta Z(t)\Big)=0.$$

We complete the proof by taking the Δ-antiderivative of the latter expression.

□

V.8 Proof of the technical Lemmas

V.8.1 Proof of Lemma V.3

Using the time scales chain rule, we obtain

$$\Delta_{\widetilde{\mathbb{T}}_s}\left(g^1_s\circ x\circ(g^0_s)^{-1}\right)(\tau)=\Delta\left(g^1_s\circ x\right)(t)\Delta_{\widetilde{\mathbb{T}}_s}\left(g^0_s\right)^{-1}(\tau).$$

Then, using the time scales derivative formula for inverse function, we obtain

$$\Delta_{\widetilde{\mathbb{T}}_s}\left(g^1_s\circ x\circ(g^0_s)^{-1}\right)(\tau)=\Delta\left(g^1_s\circ x\right)(t)\frac{1}{\Delta g^0_s(t)}. \tag{V.8.1}$$

Using the change of variable formula for time scales integrals, we obtain

$$\int_{\tau_a}^{\tau_b}L_s\left(\tau,g^1_s\circ x\circ(g^0_s)^{-1}(\tau),\Delta_{\widetilde{\mathbb{T}}_s}\left(g^1_s\circ x\circ(g^0_s)^{-1}\right)(\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau$$
$$=\int_a^b L_s\left(g^0_s(t),(g^1_s\circ x)(t),\Delta\left(g^1_s\circ x\right)(t)\frac{1}{\Delta g^0_s(t)}\right)\Delta g^0_s(t)\Delta t.$$

Finally, using the invariance condition in Equation (V.2.5), we obtain the result. □

V.8.2 Proof of Lemma V.5

For the necessary condition, let $\gamma=(t,x)\in\mathcal{F}$ be a critical point of $\mathcal{L}_{\mathbb{L}}$. Then, from Equation ($\mathrm{EL}^{\nabla\circ\Delta}$), it satisfies the following Euler–Lagrange equations

$$(\mathrm{EL}^{\nabla\circ\Delta})_{\mathbb{L}}\begin{cases}\nabla\left[\dfrac{\partial\mathbb{L}}{\partial v}(\star_\tau)\right]=\nabla\sigma(\tau)\dfrac{\partial\mathbb{L}}{\partial x}(\star_\tau),\\[2ex]\nabla\left[\dfrac{\partial\mathbb{L}}{\partial w}(\star_\tau)\right]=\nabla\sigma(\tau)\dfrac{\partial\mathbb{L}}{\partial t}(\star_\tau),\end{cases} \tag{V.8.2}$$

for all $\tau\in\mathbb{T}^\kappa_\kappa$, where $\star_\tau=(t(\tau),(x\circ t)(\tau),\Delta[t](\tau),\Delta[x\circ t](\tau))$.

By definition, we have

$$\frac{\partial \mathbb{L}}{\partial t}(\star_\tau) = \frac{\partial L}{\partial t}(\star_\tau)\Delta[t](\tau), \qquad \frac{\partial \mathbb{L}}{\partial w}(\star_\tau) = L(\star_\tau) - \Delta[x\circ t](\tau)\frac{1}{\Delta[t](\tau)}\frac{\partial L}{\partial v}(\star_\tau), \tag{V.8.3}$$

$$\frac{\partial \mathbb{L}}{\partial x}(\star_\tau) = \frac{\partial L}{\partial x}(\star_\tau)\Delta[t](\tau), \qquad \frac{\partial \mathbb{L}}{\partial v}(\star_\tau) = \frac{\partial L}{\partial v}(\star_\tau). \tag{V.8.4}$$

As $\gamma \in \mathcal{F}$, we have $(\star_\tau) = (\tau, x(\tau), \Delta x(\tau))$. As a consequence, the first Euler–Lagrange equation is equivalent to

$$\nabla\left[\frac{\partial L}{\partial v}(\star_\tau)\right] = \nabla\sigma(\tau)\frac{\partial L}{\partial x}(\star_\tau). \tag{V.8.5}$$

for all $\tau \in \mathbb{T}^\kappa_\kappa$ and the second Euler–Lagrange equation is equivalent to

$$\nabla\sigma(\tau)\frac{\partial L}{\partial t}(\star_\tau) + \nabla\left(\Delta x(\tau)\frac{\partial L}{\partial v}(\star_\tau) - L(\star_\tau)\right) = 0, \tag{V.8.6}$$

for all $\tau \in \mathbb{T}^\kappa_\kappa$, which corresponds to the condition ($\mathrm{EL}^{\mathrm{2nd}}$). As Equation (V.8.5) is the Euler–Lagrange equation associated with the Lagrangian functional $\mathcal{L}_{\Delta,\mathbb{T}}$, we obtain that x is a critical point of $\mathcal{L}_{\Delta,\mathbb{T}}$ and ($\mathrm{EL}^{\mathrm{2nd}}$) is satisfied.

For the sufficient condition, let us assume that ($\mathrm{EL}^{\mathrm{2nd}}$) is satisfied and let x be a critical point of $\mathcal{L}_{\Delta,\mathbb{T}}$ and let γ be the path such that $(t,x) \in \mathcal{F}$. The previous computations show that γ satisfies equation (V.8.5) by assumption on x and equation (V.8.6) by hypothesis. As a consequence, γ is a critical point of $\mathcal{L}_{\mathbb{L}}$. This concludes the proof. □

V.8.3 Proof of Lemma V.4

P Let $\gamma = (t,x) \in \mathcal{F}$. By definition, we have

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b \mathbb{L}\left(g_s^0(t(\tau)), (g_s^1\circ x)(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s} g_s^0(t(\tau)), \Delta_{\widetilde{\mathbb{T}}_s}\left(g_s^1\circ x\right)(t(\tau))\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{V.8.7}$$

Using the definition of $\mathbb{L}$ and the fact that $t(\tau) = \tau$ and $\Delta g_s^0(\tau) \neq 0$ for all $\tau \in \mathbb{T}^\kappa$, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b L_s\left(g_s^0(\tau), (g_s^1\circ x)(\tau), \Delta\left(g_s^1\circ x\right)(\tau)\frac{1}{\Delta g_s^0(\tau)}\right)\Delta g_s^0(\tau)\Delta\tau. \tag{V.8.8}$$

Using the invariance of $\mathcal{L}_{L,[a,b],\mathbb{T}}$ with the Lemma V.3, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b L(\tau, x(\tau), \Delta x(\tau))\,\Delta\tau. \tag{V.8.9}$$

In consequence, as $\Delta t(\tau) = 1$, we obtain

$$\mathcal{L}_{\mathbb{L}}(g_s(\gamma)) = \int_a^b \mathbb{L}(\tau, x(\tau), 1, \Delta x(\tau))\,d\tau = \mathcal{L}_{\mathbb{L}}(\gamma). \tag{V.8.10}$$

This concludes the proof. □

V.8.4 Proof of Lemma VI.4

Let $s \in \mathbb{R}$. Using the formula $g_s^0 \circ \sigma = \tilde{\sigma}_s \circ g_s^0$, we have that

$$\left[g_s^1 \circ x \circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau) = \left[g_s^1 \circ x \circ (g_s^0)^{-1} \circ \tilde{\sigma}_s \circ g_s^0\right](t) = \left[g_s^1 \circ x \circ \sigma\right](t)$$

Using the formula (V.8.1) and the change of variable formula for time scales integrals, we obtain

$$\int_{\tau_a}^{\tau_b} L_s\left(\tau, \left[g_s^1 \circ x \circ (g_s^0)^{-1}\right]^{\tilde{\sigma}_s}(\tau), \Delta_{\widetilde{\mathbb{T}}_s}\left[g_s^1 \circ x \circ (g_s^0)^{-1}\right](\tau)\right)\Delta_{\widetilde{\mathbb{T}}_s}\tau = \int_{t_a}^{t_b} L_s\left(g_s^0(t), \left[g_s^1 \circ x\right]^{\sigma}(t), \Delta\left[g_s^1 \circ x\right](t) \cdot \frac{1}{\Delta g_s^0(t)}\right)\Delta g_s^0(t)\Delta t.$$

This concludes the proof. □

V.8.5 Proof of Lemma V.9

By definition of $\mathbb{L}_\sigma$ given in equation (V.3.15), we have

$$\begin{cases} \partial_t \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_t L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right)\ w \\ \partial_x \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_x L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right)\ w \\ \partial_w \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right) - \partial_v L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right) \cdot \dfrac{v}{w} \\ & \qquad - \partial_t L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right)\ \mu(\tau)\ w \\ \partial_v \mathbb{L}_\sigma(\tau; t^\sigma, x, w, v) & = \partial_v L\left(t^\sigma - \mu(\tau)w, x, \dfrac{v}{w}\right) \end{cases} \tag{V.8.11}$$

These relations reduce over $\mathcal{F}$ as follows

$$\begin{cases} \partial_t \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_t L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \\ \partial_x \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_x L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \\ \partial_w \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) - \partial_v L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \cdot \Delta x(\tau) \\ & \qquad - \partial_t L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right)\ \mu(\tau) \\ \partial_v \mathbb{L}_\sigma\left(\tau; \tau^\sigma, x(\tau), 1, \Delta x(\tau)\right) & = \partial_v L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \end{cases} \tag{V.8.12}$$

The Euler-Lagrange equation associated to $\mathbb{L}_\sigma$ over $\mathcal{F}$ is given by

$$\begin{gathered} \Delta\left[-\mathscr{H}(t, x^\sigma, \Delta x)\right] = \partial_t L(t, x^\sigma, \Delta x), \\ \Delta\left[\partial_v L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right)\right] = \partial_x L\left(\tau, x^\sigma(\tau), \Delta x(\tau)\right) \end{gathered} \tag{V.8.13}$$

This concludes the proof. □

V.9 Conclusion and perspectives

The previous work can be generalized and applied in a variety of situations. We develop only one of them.

V.9.1 Applications to the foundations of the scale relativity

In [31], the framework of multiscale functions and scale dynamics was introduced in order to formulate rigorously some problems related to the foundations of the **scale relativity theory** developed by L. Nottale [63], [80], [81]. Multiscale functions are informally "one-parameter family of functions defined on a time scale which is variable with the parameter". These objects correspond to typical paths in a "fractal space-time" as discussed by L. Nottale [80]. The analogue of differential equations in this setting are called scale equations (see [31]). In particular, the typical paths of a fractal space-time corresponds to solution of a one-parameter family of Euler-Lagrange equations on time scales called scale Euler-Lagrange equation as for example the scale Newton equation in [31]. In this context, properties of particles can be identified with conserved quantities of the underlying Lagrangian functional on time scales (see [81, p.244]) as for example the spin (see [81, Section 6.4.3, p.288]) or the charge (see [81, Chapter 7, p.332]). Using our previous result on the Noether theorem on time scales we would like to give a full discussion of the previous statements in the context of scale equations as defined in [31]. A consequence would be a new understanding of the nature of spin and charges as coming from particular geometric nature of space-time.

Chapter VI

Noether's Theorem for Hamiltonian Systems on Time Scales

In this chapter, we prove a Noether's theorem for Hamiltonian systems on time scales. Our result is compared with previous statements obtained in the literature, in particular the one due K. Peng and Y. Luo [84] and the other one of X-H. Zhai and L. Y. Zhang [90]. Using specific examples and simulations, it is proved that these results are incorrect.

This chapter is based on the preprint "Noether's theorem for Hamiltonian systems on Time scales" *with J. Cresson, J. Palafox (UPPA) and A. Hamdouni from La Rochelle University.*

VI.1 Introduction and statement of problem

The Noether's theorem for Hamiltonian systems, even if it was already contains in the E. Noether's original formulation (see [61], §.5.5) is not so common as the one for Lagrangian equations. The main difference lies in the fact that not all the transformation groups can be considered but only canonical transformation groups which preserve the Hamiltonian character of the equations under transformations. We refer to the work of A. Mouchet [78] for a very interesting discussion of this theorem.

We are considering time scales analogues of Hamiltonian systems defined for $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ by

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p}(t, q, p), \\ \dot{p} = -\dfrac{\partial H}{\partial q}(t, q, p). \end{cases} \tag{VI.1.1}$$

Different versions of Hamiltonian systems on time scales exist. In this chapter, we consider the one introduced by C. D. Ahlbrandt, M. Bohner and J. Ridenhour in [2] which is related to the *shifted calculus of variations* as introduced by M. Bohner in [10] and a second one introduced by F. Pierret in [85] and related to the *nonshifted calculus of variations* generalizing the *discrete Hamiltonian mechanics* as defined by S. Lall and M. West [62].

A useful result by studying Hamiltonian systems is given by the Noether's theorem which gives a strong connection between group of symmetries of the functional associated to the Hamiltonian system (called variational symmetries) and first integrals of the Hamiltonian system (see the original paper of E. Noether in [61]). In particular, a variational symmetry being given, the Noether's theorem provides an explicit first integral. Our aim is then to extend the classical Hamiltonian version of the Noether's theorem on time scales.

Many results already exist in this direction with various "generalizations". Two articles are discussing exactly the same problem: the work of K. Peng and Y. Luo [84] and the one of X-H. Zhai, L. Y. Zhang [90]. Unfortunately, the Noether's theorem for Hamiltonian system on time scales (in the framework of the shifted calculus of variations) stated in these articles are incorrect.

As an example, which will be detailed in Section VI.6.1, the Hamiltonian function defined on $\mathbb{R}^4$ by

$$H(q_1, q_2, p_1, p_2) = p_1 p_2 + q_1 + q_2 \tag{VI.1.2}$$

which is considered by K. Peng and Y. Luo in [84] generates the Hamiltonian system on time scales given by

$$\begin{cases} \Delta p_1 &= -1, \\ \Delta p_2 &= -1, \\ \Delta q_1 &= p_2, \\ \Delta q_2 &= p_1, \end{cases} \tag{VI.1.3}$$

in the framework of the Δ-shifted calculus of variations on time scales. In [84], the authors consider the time scale $\mathbb{T} = \{kn,\ n \in \mathbb{N}\}$, where k is a constant. Using specific variational symmetries, the authors assert that the following two quantities

$$I_1^{PL} = p_1 + k\Delta p_1, \quad I_2^{PL} = -H - k(\Delta p_1 \Delta q_1 + \Delta p_2 \Delta q_2), \tag{VI.1.4}$$

are first integrals of the Hamiltonian systems (VI.1.3). However, a numerical implementation of the two quantities leads to the following simulations

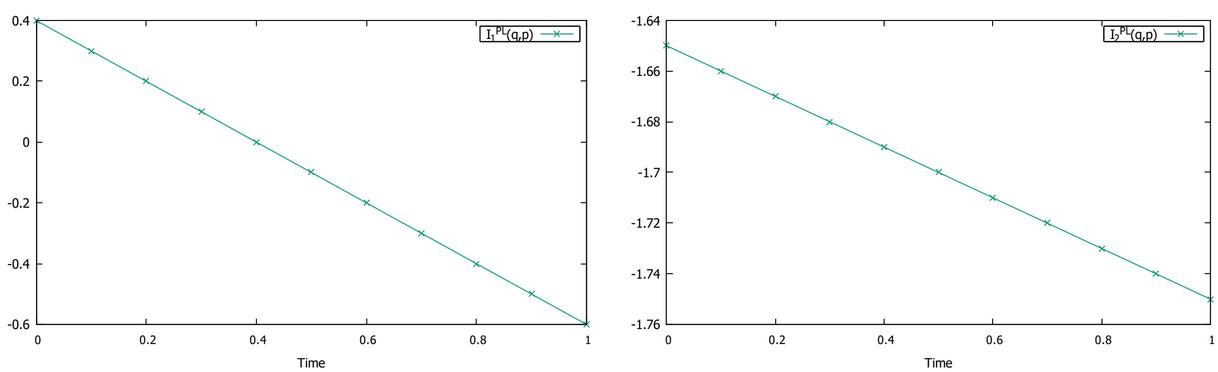


Figure VI.1: Simulations of the Peng-Luo's quantities on the solution of (VI.1.3)

which shows clearly that I_1^{PL} and I_2^{PL} are not first integrals.

The origin of the problem in these two papers can be tracked back to a previous result stated by Z. Bartosiewicz, N. Martins and D.F.M. Torres in [7] called the *second Euler-Lagrange equation* (see Eq ($\text{EL}_\sigma^{\text{2nd}}$) in Chapter V) which is implicitly or explicitly used in [84], [90]. Unfortunately, this result was proved to be incorrect in [3].

Our main goal is to prove a Hamiltonian version of the Noether's theorem on time scales in the context of the Δ-shifted and Δ-nonshifted calculus of variations. We also discuss several examples taken from the literature and perform numerical simulations. In particular, we always compare our result with the one proved in each article from which the example is taken.

Finally, in order to discuss the results of C.J. Song and Y. Zhang in [87] where they derive a Noether's theorem for Hamiltonian systems on time scales in the framework of the ∇-shifted calculus of variations, we extend our result to this situation using the Caputo duality principle. Here again, the Noether's theorem on time scales stated in [87] is proved to be wrong.

Organization of the chapter. Section VI.2 reminds definitions of Hamiltonian systems on time scales as defined by C. D. Ahlbrandt, M. Bohner, and J. Ridenhour in [2] in the framework of the Δ-shifted calculus of variations and by F. Pierret [85] in the nonshifted case. Section VI.3 introduce the notion of admissible canonical group of transformations and in Section VI.3.1 we derive the invariance of the Hamiltonian functional on time scales. In Section VI.4, our main results are stated and proved, i.e., the Noether's theorem on time scales. Section VI.5 gives the dual result in the framework of the ∇-shifted calculus of variations. Finally, in Section VI.6 we discuss several examples and we provide numerical simulations.

VI.2 Remainder about Hamiltonian systems on time scales

In this section, definitions about *Hamiltonian systems on time scales* are presented in the framework of the shifted [2] and nonshifted calculus of variations on time scales [85].

VI.2.1 The shifted case

C. D. Ahlbrandt, M. Bohner, and J. Ridenhour in [2] introduced a notion of Hamiltonian systems on time scales:

Definition VI.1. *Let $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ be a function of class $\mathscr{C}^2$ in each of its variables. Let $\mathbb{T}$ be a time scales. The Hamiltonian system associated to H on $\mathbb{T}$ is defined by*

$$\begin{cases} \Delta q = \dfrac{\partial H}{\partial p}(t, q^\sigma, p), \\ \Delta p = -\dfrac{\partial H}{\partial q}(t, q^\sigma, p). \end{cases} \tag{VI.2.1}$$

Using the shifted calculus of variations on time scales developed in [10], M. Bohner proved that the previous Hamiltonian systems on time scales can be obtained as critical points of shifted Lagrangian functionals on time scales. Precisely, we have:

Theorem VI.1. *The solutions of the Hamiltonian system (IX.5.1) on $\mathbb{T}$ correspond to critical points of the time scales functional*

$$\mathcal{L}^\sigma_{H,[a,b]_\mathbb{T}}(q, p) = \int_a^b [p \cdot \Delta q - H(t, q^\sigma, p)] \, \Delta t. \tag{$\mathcal{L}^\sigma_H$}$$

As an example, one can consider the Hamiltonian system defined by

$$H(t, q_1^\sigma, q_2^\sigma, p_1, p_2) = p_1 p_2 + q_1^\sigma + q_2^\sigma, \tag{VI.2.2}$$

which gives the following differential equation

$$\begin{cases} \Delta q_1 = p_2, \\ \Delta q_2 = p_1, \\ \Delta p_1 = -1, \\ \Delta p_2 = -1. \end{cases} \tag{VI.2.3}$$

VI.2.2 The nonshifted case

F. Pierret introduced in [85] a notion of Hamiltonian systems on time scales adapted to the framework of the nonshifted calculus of variations. Precisely, we have:

Definition VI.2. *Let $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ be a function of class $\mathscr{C}^2$ in each of its variables. Let $\mathbb{T}$ be a time scales and assume that σ is ∇-differentiable on $\mathbb{T}_\kappa$. The Hamiltonian system associated to H on $\mathbb{T}$ is defined by*

$$\begin{cases} \Delta q = \dfrac{\partial H}{\partial p}(t, q, p), \\ \nabla p = -\nabla\sigma \dfrac{\partial H}{\partial q} H(t, q, p). \end{cases} \tag{VI.2.4}$$

Here again, one can prove that Hamiltonian systems are critical point of Lagrangian functionals on time scales:

Theorem VI.2. *The solutions of the Hamiltonian systems (IX.5.2) on $\mathbb{T}$ correspond to critical points of the time scales functional*

$$\mathcal{L}_{H,[a,b]_\mathbb{T}}(q, p) = \int_a^b [p \cdot \Delta q - H(t, q, p)]\, \Delta t. \tag{$\mathcal{L}_H$}$$

We refer to the work of F. Pierret [85] for more details.

Using canonical transformation groups, we can defined **canonical variational symmetries** of Hamiltonian systems on time scales.

VI.3 Admissible canonical transformations group

Working with time scales imposes some restrictions on the transformation groups that one can consider. In the following, we need the notion of $(\Delta, \mathbb{T})$**-admissible projectable group of transformations**:

Definition VI.3 ($(\Delta, \mathbb{T})$-admissible projectable group of canonical transformations)**.** *A projectable group of canonical transformations $\{\phi_s\}_{s\in\mathbb{R}}$ is called a $(\Delta, \mathbb{T})$-admissible projectable group of canonical transformations if for all $s \in \mathbb{R}$, the function ϕ_s^0 verifies:*

- *ϕ_s^0 is strictly increasing,*
- *$\Delta\phi_s^0 \neq 0$ and $\Delta\phi_s^0$ is rd-continuous and such that,*
- *the set defined by $\widetilde{\mathbb{T}}_s = \phi_s^0(\mathbb{T})$ is a time scales.*
- *$\Delta_{\widetilde{\mathbb{T}}_s} \left(\phi_s^0\right)^{-1}$ exists.*

A notion of $(\nabla, \mathbb{T})$-admissible projectable group of canonical transformations can be defined in the same way.

VI.3.1 Invariance of a Hamiltonian functional on time scales

VI.3.1.1 The shifted case

Definition VI.4. *The canonical transformation group (IV.5.1) is a variational symmetry of the Hamiltonian functional ($\mathcal{L}^{\sigma}_{H}$) if*

$$\mathcal{L}^{\sigma}_{H,[t_a,t_b]_{\mathbb{T}}}(q,p) = \mathcal{L}^{\sigma_s}_{H_s,[\tau_a,\tau_b]_{\widetilde{\mathbb{T}}_s}}(q_s,p_s), \tag{VI.3.1}$$

for any subinterval $[t_a,t_b] \subseteq [a,b]$, where H_s is the Hamiltonian function associated to the differential equation in the new variables.

An explicit form of the previous invariance relation (VI.3.1) is given by

$$\int_{t_a}^{t_b} \Big[p\cdot\Delta q - H(t,q^\sigma,p)\Big]\Delta t = \int_{\tau_a}^{\tau_b} \Big[\left(\phi^2_s\circ p\circ(\phi^0_s)^{-1}\right)\cdot\Delta_{\widetilde{\mathbb{T}}_s}\left(\phi^1_s\circ q\circ(\phi^0_s)^{-1}\right) \\ - H_s\left(\tau,\left[\phi^1_s\circ q\circ(\phi^0_s)^{-1}\right]^{\sigma_s},\phi^2_s\circ p\circ(\phi^0_s)^{-1}\right)\Big]\Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{VI.3.2}$$

Performing the change of variable $\tau = \phi^0_s(t)$, we obtain a more tractable form of the invariance condition:

Lemma VI.1. *The invariance condition (VI.3.1) is equivalent to*

$$\mathcal{L}^{\sigma}_{H,[t_a,t_b]_{\mathbb{T}}}(q,p) = \int_{t_a}^{t_b} \Big[(\phi^2_s(p)\cdot\Delta\left(\phi^1_s(q)\right) - H_s\left(\phi^0_s(t),\left[\phi^1_s(q)\right]^\sigma,\phi^2_s(p)\right)\Delta(\phi^0_s)\Big]\Delta t. \tag{VI.3.3}$$

Deriving the previous equality with respect to s and taking $s=0$, we obtain:

Lemma VI.2. *The Hamiltonian functional ($\mathcal{L}^{\sigma}_{H}$) is invariant under the canonical group of transformation (IV.5.1) if*

$$-\frac{\partial G}{\partial q}(\star)\cdot\Delta q + p\cdot\Delta\left(\frac{\partial G}{\partial p}(\star)\right) - \left[\frac{\partial H}{\partial t}(t,\star)\zeta + \frac{\partial H}{\partial q}(t,\star)\cdot\left(\frac{\partial G}{\partial p}(\star)\right)^\sigma - \frac{\partial H}{\partial p}(t,\star)\cdot\frac{\partial G}{\partial q}(\star)\right] - H(t,\star)\Delta\zeta = 0,$$

where $\star = (q^\sigma,p)$.

VI.3.1.2 The nonshifted case

Following the same computations as in Subsection VI.3.1

Definition VI.5. *The Hamiltonian functional ($\mathcal{L}_H$) is invariant under the canonical transformation group (IV.5.1) if*

$$\mathcal{L}_{H,[t_a,t_b]_{\mathbb{T}}}(q,p) = \int_{\tau_a}^{\tau_b} \Big[\left(\phi^1_s\circ p\circ(\phi^0_s)^{-1}\right)\cdot\Delta_{\widetilde{\mathbb{T}}_s}\left(\phi^2_s\circ q\circ(\phi^0_s)^{-1}\right) \\ - H_s\left(\tau,\phi^2_s\circ q\circ(\phi^0_s)^{-1},\phi^1_s\circ p\circ(\phi^0_s)^{-1}\right)\Big]\Delta_{\widetilde{\mathbb{T}}_s}\tau. \tag{VI.3.4}$$

Lemma VI.3. *The invariance condition (VI.3.4) is equivalent to*

$$\mathcal{L}_{H,[t_a,t_b]_{\mathbb{T}}}(q,p) = \int_{t_a}^{t_b} \Big[(\phi^1_s(p)\cdot\Delta\left(\phi^2_s(q)\right) - H_s\left(\phi^0_s(t),\phi^1_s(q),\phi^2_s(p)\right)\Delta(\phi^0_s)\Big]\Delta t. \tag{VI.3.5}$$

Lemma VI.4. *The Hamiltonian functional ($\mathcal{L}_H$) is invariant under the canonical group of transformation (IV.5.1) if*

$$-\frac{\partial G}{\partial q}(\star)\cdot\Delta q + p\cdot\Delta\left(\frac{\partial G}{\partial p}(\star)\right) - \left[\frac{\partial H}{\partial t}(t,\star)\zeta + \frac{\partial H}{\partial q}(t,\star)\cdot\left(\frac{\partial G}{\partial p}(\star)\right) - \frac{\partial H}{\partial p}(t,\star)\cdot\frac{\partial G}{\partial q}(\star)\right] - H(t,\star)\Delta\zeta = 0,$$

where $\star = (q,p)$.

VI.4 Noether's theorem for Hamiltonian systems on time scales

We are now ready to state the time scales version of the Noether's theorem for Hamiltonian systems.

VI.4.1 Noether's theorem - shifted case

Theorem VI.3 (Nother's theorem for Hamiltonian systems on time scales)**.** *If the Hamiltonian functional* ($\mathcal{L}_H^\sigma$) *is invariant under the canonical variational symmetry* (IV.5.1) *then a first integral is given by*

$$I(q,p) = p \cdot \frac{\partial G}{\partial q}(\star) - H(t,\star)\zeta - \int_a^t \zeta \left[\frac{\partial H}{\partial t}(t,\star) - \Delta\left(H(t,\star)\right)\right] \Delta t + \int_a^t \mu(t)\Delta\zeta\Delta\left(H(t,\star)\right) \Delta t,$$

meaning that

$$\Delta\left[I(q,p)\right] = 0, \tag{VI.4.1}$$

over the solutions of the Hamiltonian system (IX.5.1).

Of course, we recover the classical Noether's theorem for Hamiltonian systems when $\mathbb{T} = [a,b]$. Indeed, in this case, we have $\mu(t) = 0$ which cancels the last term and moreover, we have the second Euler-Lagrange equation

$$\frac{dH}{dt}(t,q,p) - \frac{\partial H}{\partial t}(t,q,p) = 0. \tag{VI.4.2}$$

which cancels the second term. As a consequence, when $\mathbb{T} = [a,b]$, a first integral is given by

$$p\frac{\partial G}{\partial q}(q,p) - H(t,q,p)\,\zeta. \tag{VI.4.3}$$

Proof. Using Lemma VI.2 and the Hamiltonian system (IX.5.1), we have that

$$p \cdot \Delta\left(\frac{\partial G}{\partial p}\right) - \frac{\partial H}{\partial q} \cdot \left(\frac{\partial G}{\partial p}\right)^\sigma = p \cdot \Delta\left(\frac{\partial G}{\partial p}\right) + \Delta p \cdot \left(\frac{\partial G}{\partial p}\right)^\sigma, \tag{VI.4.4}$$

which reduces to

$$\Delta\left(p \cdot \frac{\partial G}{\partial p}\right), \tag{VI.4.5}$$

thanks to the Leibniz formula. Moreover, we have

$$\frac{\partial G}{\partial q} \cdot \Delta q - \frac{\partial H}{\partial p} \cdot \frac{\partial G}{\partial q} = \frac{\partial G}{\partial q} \cdot \Delta q - \Delta q \cdot \frac{\partial G}{\partial q} = 0. \tag{VI.4.6}$$

As a consequence, the invariance relation (VI.2) reduces to

$$\Delta\left(p \cdot \frac{\partial G}{\partial p}\right) - \frac{\partial H}{\partial t}\,\zeta - H\,\Delta\zeta = 0. \tag{VI.4.7}$$

As we have

$$\Delta(H\zeta) = \Delta H\,\zeta^\sigma + H\,\Delta\zeta, \tag{VI.4.8}$$

we obtain

$$\Delta\left(p \cdot \frac{\partial G}{\partial p} - H\zeta\right) - \frac{\partial H}{\partial t}\zeta + \Delta H\,\zeta^\sigma = 0. \tag{VI.4.9}$$

In order to be as close as possible to the continuous case, we use the relation

$$\zeta^\sigma = \zeta + \mu(t)\Delta\zeta, \tag{VI.4.10}$$

to obtain the final form

$$\Delta\left(p \cdot \frac{\partial G}{\partial p} - H\zeta\right) + \zeta\left(\Delta H - \frac{\partial H}{\partial t}\right) + \mu(t)\Delta H\, \Delta\zeta = 0. \tag{VI.4.11}$$

This concludes the proof by taking a Δ-antiderivative. □

VI.4.2 Noether's theorem - nonshifted case

Theorem VI.4 (Nother's theorem for Hamiltonian systems on time scales - nonshifted case)**.** *Let σ is ∇-differentiable on $\mathbb{T}_\kappa$. If the Hamiltonian functional ($\mathcal{L}_H$) is invariant under the canonical variational symmetry (IV.5.1), then a first integral is given by*

$$I(q,p) = p \cdot (\partial_q G)^\sigma - H(t,\star) \cdot \zeta^\sigma - \int_a^t \zeta\Big[\nabla\sigma\, \partial_t H(t,\star) - \nabla\left(H(t,\star)\right)\Big]\nabla t, \tag{VI.4.12}$$

meaning that

$$\nabla\left[I(q,p)\right] = 0, \tag{VI.4.13}$$

over the solutions of the Hamiltonian system (IX.5.2).

Proof. Assume that σ is ∇-differentiable on $\mathbb{T}_\kappa$. Multiplying the equation (VI.4) by $\nabla\sigma$ gives

$$p \cdot \nabla\sigma\Delta(\partial_p G(\star)) - \nabla\sigma\partial_t H(t,\star)\zeta - \nabla\sigma\partial_q H(t,\star) \cdot \partial_p G(\star) - H(t,\star)\nabla\sigma\Delta\zeta = 0, \tag{VI.4.14}$$

Over the solution of (IX.5.2), we have that $\nabla p = -\nabla\sigma\, \partial_q H$ and with the help of Leibniz formula, the equation (VI.4.14) reduces to

$$\nabla\Big[p \cdot (\partial_p G(\star))^\sigma\Big] - \nabla\sigma\, \partial_t H(t,\star)\zeta - H(t,\star)\nabla\zeta^\sigma = 0. \tag{VI.4.15}$$

Again, applying Leibniz formula to the last term, the previous equation can be written as

$$\nabla\Big[p \cdot (\partial_p G(\star))^\sigma - H(t,\star)\zeta^\sigma\Big] - \nabla\sigma\, \partial_t H(t,\star)\zeta + \nabla H(t,\star)\zeta = 0. \tag{VI.4.16}$$

The proof is completed by taking the ∇-antiderivative of this latter equation. □

VI.5 The (∇, ρ)-version of Noether's theorem on time scales

We give a (∇, ρ)-version of the Noether's theorem on time scales as discussed for example by C-J Song and Y. Zhang in [87, Theorem 4 p.28]. Our result is deduced from the (Δ, σ) Noether's theorem on time scales using the **Caputo duality principle** first introduced by C. Caputo in [19] (see Section V.6.1 in Chapter V)

VI.5.1 $\nabla \circ \nabla$-Hamiltonian system

In [87], the duality principle was used to define Hamiltonian systems on time scales with the functional of the form

$$\mathcal{L}^{\rho}_{H,a,b,\mathbb{T}}(q,p) = \int_a^b \left[p \cdot \nabla q - H(t, q^{\rho}, p)\right] \nabla t, \qquad (\mathcal{L}^{\rho}_H)$$

they proved that the critical points of the time scales functional are solutions of the following Hamiltonian system

$$\begin{cases} \nabla q = \dfrac{\partial H}{\partial p}(t, q^{\rho}, p), \\ \nabla p = -\dfrac{\partial H}{\partial q} H(t, q^{\rho}, p). \end{cases} \qquad \text{(VI.5.1)}$$

We can now establish our time scales Noether theorem using duality principle which given in [87]. Precisely, we have:

Theorem VI.5 ((∇, ρ) - Nother's theorem on time scales)**.** *If the Hamiltonian functional* ($\mathcal{L}^{\rho}_H$) *is invariant under the canonical variational symmetry* (IV.5.1) *then a first integral is given by*

$$I(q,p) = p \cdot \partial_q G(\star) - H(t, \star)\,\zeta - \int_a^t \zeta\left[\partial_t H(t, \star) - \nabla\left(H(t, \star)\right)\right]\nabla t - \int_a^t \nu(t) \nabla\zeta \nabla\left(H(t, \star)\right)\nabla t, \qquad \text{(VI.5.2)}$$

meaning that

$$\nabla\left[I(q,p)\right] = 0, \qquad \text{(VI.5.3)}$$

over the solutions of the Hamiltonian system (IX.5.1)*.*

The proof follows essentially the same steps as in the Δ case and without additional difficulties. As a consequence, we let a detailed proof to the reader.

VI.6 Examples and simulations

In this Section, we discuss several examples given in [84], [87], [90] where Noether's theorem on time scales is derived. We implement the constant of motion that these authors have obtained and show that they do not remain constant on the solutions of the associated Hamiltonian system on time scales. We also provide simulations for the constant of motion derived using our Noether's theorem. Note that these results always fail for transformation groups for which a transformation in time is needed. The main reason is that these authors use an incorrect result of Z. Bartosiewicz and D.F.T. Torres [8] called the second order Euler-Lagrange equation, which is interpreted in the Hamiltonian setting as

$$\Delta\left(H - \mu(t)\frac{\partial H}{\partial t}\right) = \frac{\partial H}{\partial t}. \qquad \text{(VI.6.1)}$$

In the continuous case, the previous relation relates the total derivative with respect to t of $H(q(t), p(t), t)$ and the partial derivative of H evaluated on the solution, namely $\partial_t H(q(t), p(t), t)$. As a consequence, when the Hamiltonian function is autonomous, i.e., does not depend on time, the Hamiltonian function itself is a constant of motion. Unfortunately, this relation is not preserved in the Hamiltonian setting.

VI.6.1 An example of K. Peng and Y. Luo

Symmetries and conservation laws. We consider the Hamiltonian function

$$H(t, q_1, q_2, p_1, p_2) = p_1 p_2 + q_1^\sigma + q_2^\sigma, \tag{VI.6.2}$$

which generates the time scale Hamiltonian system

$$\begin{cases} \Delta q_1 = p_2, \\ \Delta q_2 = p_1, \\ \Delta p_1 = -1, \\ \Delta p_2 = -1. \end{cases} \tag{VI.6.3}$$

This system is studied by K. Peng and Y. Luo in [84] over the time scale

$$\mathbb{T} = \{kn,\ n \in \mathbb{N}\}, \quad k \text{ is a constant.} \tag{VI.6.4}$$

We have two natural canonical variational symmetries. The first one, is associated to the following translation group generated by

$$\zeta = 0,\ G(q,p) = p_1 - p_2, \tag{VI.6.5}$$

which gives the following family of transformations

$$\phi_s^0(t) = t,\ \phi_s^1(q,p) = (q_1 + s, q_2 - s) + o(s^2),\ \phi_s^2(q,p) = p. \tag{VI.6.6}$$

Using our Noether's theorem for Hamiltonian systems on time scales we obtain the following first integral

$$I_1(q,p) = p_1 - p_2. \tag{VI.6.7}$$

This can be directly checked by computing

$$\Delta[p_1 - p_2] = \Delta[p_1] - \Delta[p_2] = -1 + 1 = 0. \tag{VI.6.8}$$

A second first integral can be obtain using the independence of the Hamiltonian with respect to time, i.e., considering transformations such that

$$G = 0,\ \zeta = 1. \tag{VI.6.9}$$

We need to compute $\Delta\left[H(t, q_1^\sigma, q_2^\sigma, p_1, p_2)\right]$.

Lemma VI.5. *For an arbitrary time scale, we have*

$$\Delta H = \mu - 2\mu^\sigma + (p_1 + p_2)\left(\frac{\mu^\sigma}{\mu} - 1\right). \tag{VI.6.10}$$

Proof. Using the Leibniz formula, we obtain

$$\begin{aligned} \Delta H &= \Delta p_1 p_2^\sigma + p_1 \Delta p_2 + \Delta(q_1^\sigma) + \Delta(q_2^\sigma), \\ &= -p_2^\sigma - p_1 + \Delta(q_1^\sigma) + \Delta(q_2^\sigma). \end{aligned} \tag{VI.6.11}$$

Moreover, we have

$$p_2^\sigma = \mu \Delta p_2 + p_2 = -\mu + p_2, \tag{VI.6.12}$$

and

$$\Delta(q^\sigma) = \frac{\mu^\sigma}{\mu} (\Delta q)^\sigma . \tag{VI.6.13}$$

This gives

$$\Delta(q_1^\sigma) = \frac{\mu^\sigma}{\mu} p_2^\sigma \text{ and } \Delta(q_2^\sigma) = \frac{\mu^\sigma}{\mu} p_1^\sigma . \tag{VI.6.14}$$

As a consequence, we obtain

$$\Delta H = \mu - p_2 - p_1 + \frac{\mu^\sigma}{\mu} (p_1^\sigma + p_2^\sigma) . \tag{VI.6.15}$$

Using again relation (VI.6.12) and its analogue for p_1^σ, one has

$$\Delta H = \mu - 2\mu^\sigma + (p_1 + p_2) \left(\frac{\mu^\sigma}{\mu} - 1 \right) . \tag{VI.6.16}$$

This concludes the proof. □

On the time scale $\mathbb{T} = \{kn; n \in \mathbb{N}\}$, one has

$$\mu(t) = k. \tag{VI.6.17}$$

As a consequence, $\mu^\sigma = \mu = k$, $\mu^\sigma/\mu = 1$ and ΔH reduces to

$$\Delta H = -k. \tag{VI.6.18}$$

Using (VI.6.18) and the fact that $\partial_t H = 0$, our Noether's theorem for Hamiltonian systems on time scales gives the first integral

$$I_2(q, p) = -H - kt. \tag{VI.6.19}$$

This can be also verified directly by computing ΔI_2. Indeed, we have

$$\Delta I_2 = -\Delta H - k\Delta t = k - k = 0. \tag{VI.6.20}$$

Numerical test. K. Peng and Y. Luo give two first integrals for the previous Hamiltonian system which are given by

$$I_1^{PL} = p_1 + k\Delta p_1, \quad I_2^{PL} = -H - k (\Delta p_1 \, \Delta q_1 + \Delta p_2 \, \Delta q_2) . \tag{VI.6.21}$$

These two quantities are *not* first integrals of the Hamiltonian system (VI.6.3) on the time scale (VI.6.4). Indeed, over the solutions of the Hamiltonian system we have

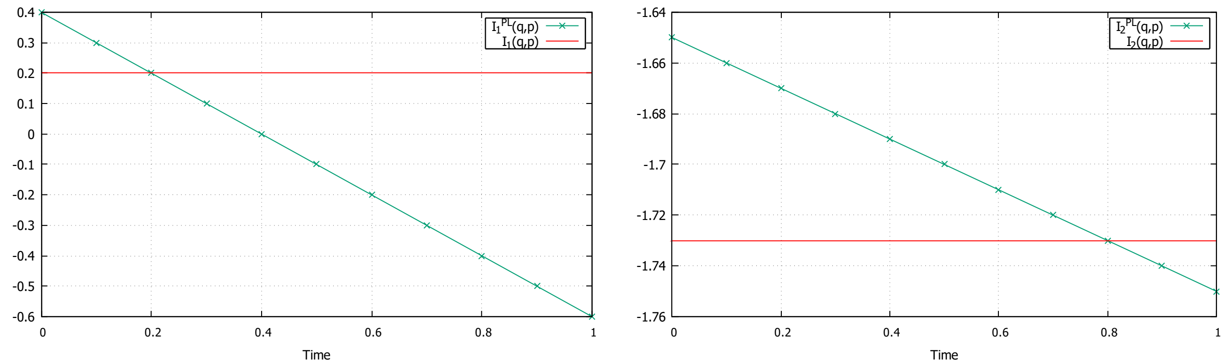
$$I_1^{PL} = p_1 - k, \quad I_2^{PL} = -H + k(p_1 + p_2). \tag{VI.6.22}$$

As a consequence, we obtain

$$\Delta I_1^{PL} = \Delta p_1 = -1, \quad \Delta I_2^{PL} = -\Delta H + k(\Delta p_1 + \Delta p_2) = k - 2k = -k. \tag{VI.6.23}$$

As we can see, we have $\Delta I_1^{PL} \neq 0$ and $\Delta I_2^{PL} \neq 0$ which contradicts the results of K. Peng and Y. Luo in ([84], Section VI).

We provide some implementations of the previous results which confirm that the quantities derived by K. Peng and Y. Luo are not first integrals of the Hamiltonian system. We use the following initial condition $p_1(0) = 0.5$, $p_2(0) = 0.3$, $q_1(0) = 1$, $q_2(0) = 0.5$.



(a) The quantities I_1 and I_1^{PL}

(b) The quantities I_2 and I_2^{PL}

Figure VI.2: Simulations of the Peng-Luo's quantities and our first integrals

VI.6.2 Two examples of X-H. Zhai, L. Y. Zhang

We discuss two examples given by X-H. Zhai and L.Y. Zhang in [90] where a Noether's theorem for Hamiltonian systems on time scales is formulated (Theorem 4, p.7). We describe the first one (Example 2, p.8) for which they provide an exact first integral and the Kepler problem on time scales (Example 1, p.6) which is also Hamiltonian and for which there result implies that the Hamiltonian itself is a first integral (see [90] after equation (61)). This result is not correct and this invalidate the Theorem proved in [90].

VI.6.2.1 The linear vibration system and The "multiplier problem" on time scales

In [90], the authors consider a special case of the second order time scale equation

$$\Delta \circ \Delta q + \delta \left(\Delta q\right)^{\sigma} . \tag{VI.6.24}$$

where δ is a real constant called the **linear vibration system** on time scales.

This equation by itself can not be obtained as an Euler-Lagrange equation due to the linear term in $(\Delta(q))^{\sigma}$. However, one can use an artifice which enable use to transform this equation in variational form. The trick goes back to two articles of A. Hirsch ([54], [55]) about the **Helmholtz's conditions** for second order differential equation (see [83, p.378]) and where he considers the "**multiplier problem**" (see [83, p.378]): *when can multiply a differential equation by a differential function so as to make it an Euler-Lagrange equation?* Generalizing this question in the time scales setting, we consider the following problem: find a Δ differentiable function f such that f multiply by the previous equation is an Euler-Lagrange equation on time scales.

Let $\gamma : \mathbb{T} \to \mathbb{R}$ be a function such that γ is strictly positive on $\mathbb{T}$. Then, equation (VI.6.24) is equivalent to

$$\gamma \cdot \left(\Delta \circ \Delta q + \delta \left(\Delta q\right)^{\sigma}\right) = 0. \tag{VI.6.25}$$

This new equation possess a Lagrangian form. Indeed, we have for an arbitrary Δ-differentiable function $f : \mathbb{T} \to \mathbb{R}$ the relation

$$\Delta(f\,\Delta q) = \Delta f\,(\Delta q)^{\sigma} + f(\Delta \circ \Delta q). \tag{VI.6.26}$$

As a consequence if f is such that $\Delta f = \delta f$, one obtains

$$\Delta(f\,\Delta q) = f\big(\delta\,(\Delta q)^{\sigma} + \Delta \circ \Delta q\big). \tag{VI.6.27}$$

If the time scale $\mathbb{T}$ is such that $\mu(t) = \mu$ for all $t \in \mathbb{T}$, a solution of the equation $\Delta f = \delta f$ is well known when

$$\delta = \frac{e^{\gamma\mu} - 1}{\mu}, \tag{VI.6.28}$$

for some real constant γ. Indeed, we have:

Lemma VI.6. *Let $\mathbb{T}$ be a time scale such that $\mu(t) = \mu$ is a constant, then the solution of the time scale equation*

$$\Delta f = \frac{e^{\gamma\mu} - 1}{\mu} f, \tag{VI.6.29}$$

is given by $f(t) = e^{\gamma t}$ for all $t \in \mathbb{T}$ up to a real multiplicative constant.

As $e^{\gamma t}$ is strictly positive for all $t \in \mathbb{T}$, we deduce that equation (VI.6.27) with δ given by (VI.6.28) is equivalent to

$$\frac{e^{\gamma\mu} - 1}{\mu}\,(\Delta q)^{\sigma} + \Delta \circ \Delta q = 0. \tag{VI.6.30}$$

We deduce the following result:

Lemma VI.7. *Let γ be a real constant and $\mathbb{T}$ be a time scale such that $\mu(t) = \mu$ is a constant over $\mathbb{T}$. The second order time scale equation on $\mathbb{T}$*

$$\frac{e^{\gamma\mu} - 1}{\mu}\,(\Delta q)^{\sigma} + \Delta \circ \Delta q = 0. \tag{VI.6.31}$$

is equivalent to the Euler-Lagrange equation associated to

$$L(t, q, v) = \frac{1}{2} e^{\gamma t} v^2, \tag{VI.6.32}$$

over $\mathbb{T}$ in the Δ-shifted calculus of variations setting.

A simple computation leads to the following result:

Lemma VI.8. *The Hamiltonian system on time scale associated to the Lagrangian (VI.6.32) is given by*

$$H(t, q, p) = \frac{1}{2} e^{-\gamma t} p^2. \tag{VI.6.33}$$

The time scale Hamiltonian system associated to H is given by

$$\begin{cases} \Delta q = e^{-\gamma t} p, \\ \Delta p = 0. \end{cases} \tag{VI.6.34}$$

As the Hamiltonian does not depend on the variable q we have trivially that p is a first integral of the Hamiltonian system. We can recover this result by taking $G = p$ and $\zeta = 0$, i.e., considering the invariance of H under the translation in q. Indeed, in this case our Theorem gives as a first integral $I_1 = p\partial_p G = p$.

In order to find more general symmetries one can use the invariance relation (VI.2) which reduces to

$$p\,\Delta\left(\frac{\partial G}{\partial p}\right) - H\left(-\gamma\zeta + \Delta\zeta\right) = 0. \tag{VI.6.35}$$

Taking $G = 0$, equation (VI.6.35) imposes that ζ satisfies $\Delta\zeta = \gamma\zeta$ whose solution is given by $\zeta = e^{c(\gamma)t}$, where

$$c(\gamma) = \frac{\ln(\gamma\mu + 1)}{\mu}. \tag{VI.6.36}$$

In order to apply the Noether's theorem we compute ΔH.

Lemma VI.9. *We have $\Delta H(t, q(t), p(t))\tilde{c}(-\gamma)H(t, q(t), p(t))$ where $(q(t), p(t))$ is a solution of the time scale Hamiltonian system (VI.6.34) where $\tilde{c}(-\gamma)$ is given by*

$$\tilde{c}(-\gamma) = \frac{e^{-\gamma\mu} - 1}{\mu}. \tag{VI.6.37}$$

As a consequence, the quantity $\Delta H - \partial_t H$ is equal to $H(t, q(t), p(t))\left(\tilde{c}(-\gamma) + \gamma\right)$. Moreover, as $\Delta p = 0$ we deduce that p is a constant. As a consequence, we have

$$\int_a^t \zeta\left(\Delta H - \frac{\partial H}{\partial t}\right)\Delta t = (\tilde{c}(-\gamma) + \gamma)\frac{p^2}{2}\int_a^t e^{-\gamma t}\zeta\Delta t, \tag{VI.6.38}$$

and

$$\int_a^t \mu\Delta\zeta\Delta H\Delta t = \mu(\tilde{c}(\gamma) + \gamma)\frac{p^2}{2}\int_a^t e^{-\gamma t}\Delta\zeta\Delta t. \tag{VI.6.39}$$

As ζ satisfies $\Delta\zeta = \gamma\zeta$, we have finally

$$\int_a^t \zeta\left(\Delta H - \frac{\partial H}{\partial t}\right)\Delta t + \int_a^t \mu\Delta\zeta\Delta H\Delta t = (\tilde{c}(\gamma) + \gamma)\frac{p^2}{2}(1 + \mu\gamma)\int_a^t e^{-\gamma t}\zeta\Delta t. \tag{VI.6.40}$$

This means that we recover again the fact that p is a first integral as waited for.

A more interesting symmetry can be obtain by assuming that G is of the form $G = \alpha pq$ where α is a constant. In this case, the invariance relation leads to

$$2\alpha + \gamma\zeta - \Delta\zeta = 0. \tag{VI.6.41}$$

Taking $\zeta = 2$, one obtain $\alpha = -\gamma$ and the following first integral:

$$I = -\gamma pq + p^2\left(-e^{-\gamma t} + (\tilde{c}(\gamma) + \gamma)\int_a^t e^{-\gamma t}\Delta t\right). \tag{VI.6.42}$$

As p is also a first integral, the previous relation reduced to

$$\tilde{I} = -\gamma q + p\left(-e^{-\gamma t} + (\tilde{c}(-\gamma) + \gamma)\int_a^t e^{-\gamma t}\Delta t\right). \tag{VI.6.43}$$

Finally, using the fact that

$$\int_a^t e^{-\gamma t}\Delta t = \frac{1}{\tilde{c}(-\gamma)}e^{-\gamma t}, \tag{VI.6.44}$$

we obtain the following result:

Lemma VI.10. *The Hamiltonian functional is invariant under the canonical transformation groups generated by $G = -\gamma pq$ and $\zeta = 2$. The first integral is given by*

$$I = -\tilde{c}(-\gamma)q + e^{-\gamma t}p. \tag{VI.6.45}$$

This first integral coincides with the one given by X.H. Zhai and L.Y. Zhang in [90, Example 2].
The simulation of the first integral (VI.6.45) on $\mathbb{T} = h\mathbb{Z}$ with $h = 0.2$ and let the initial condition $p(0) = 0.5$, $q(0) = 0.3$.

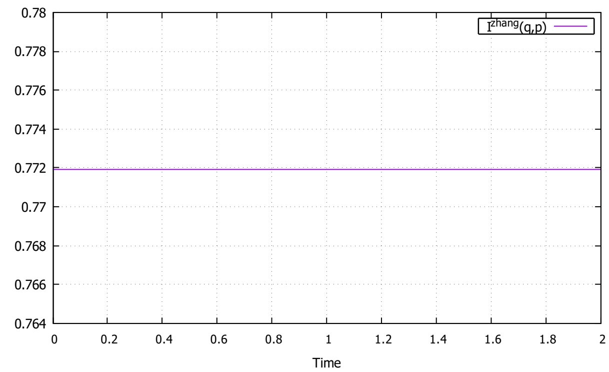


Figure VI.3: Simulation of the quantity (VI.6.45) on time scales $\mathbb{T}$

VI.6.3 A time scales nonshifted example

Consider the following Lagrangian [36]

$$L(q, v) = \frac{1}{2}\left(v^2 - \frac{1}{q^2}\right), \quad (q, v) \in \mathbb{R}^* \times \mathbb{R}. \tag{VI.6.46}$$

The Hamiltonian associate to L is given by

$$H(q, p) = p\Delta q - L(q, v) = \frac{1}{2}\left(p^2 + \frac{1}{q^2}\right). \tag{VI.6.47}$$

The corresponding Hamiltonian system over time scales $\mathbb{T} = \{\frac{n}{3}, \quad n \in \mathbb{N}\}$ is given by

$$\begin{cases} \Delta q = p, \\ \nabla p = \dfrac{1}{q^3} \end{cases} \tag{VI.6.48}$$

Let us look for some symmetries using the equation (VI.4). Since $\partial_t H = 0$, then if we take $\xi = 1$ and $G = 0$, the equation (VI.4) is satisfied and therefore we have the following family of transformations

$$\phi_s^0(t) = t + s, \quad \phi_s^1(q, p) = q, \quad \phi_s^2(q, p) = p. \tag{VI.6.49}$$

So that, Noether theorem gives the following first integral

$$I_1(q, p) = -H(p(t), q(t)) + \int_0^t \nabla H(p(t), q(t)) \nabla t = H(p(0), q(0)). \tag{VI.6.50}$$

Now, let $G(q, p) = pq$, then the equation (VI.4) reduces to

$$p^2 + \frac{1}{q^2} - \frac{1}{2}\left(p^2 + \frac{1}{q^2}\right)\Delta\zeta = 0. \tag{VI.6.51}$$

If we take $\xi = 2t$, the previous equation satisfies, then we have the following scaling groups

$$\phi_s^0(t) = e^{2s}t, \quad \phi_s^1(q,p) = e^s q, \quad \phi_s^2(q,p) = e^{-s}p. \tag{VI.6.52}$$

Using Noether theorem gives the following first integral

$$I_2(p,q) = p(t)\cdot q^\sigma(t) - \sigma(t)\ H\left(p(t), q(t)\right) + \int_0^t t\ \nabla\ H\left(p(t), q(t)\right)\ \nabla t. \tag{VI.6.53}$$

Of course, in the continuous case, the Hamiltonian $p^2 + 1/q^2$ is conserved so that the integral term vanishes and the previous quantity becomes

$$I_2(p,q) = p(t)\cdot q(t) - t\left(p^2(t) + \frac{1}{q^2(t)}\right),$$

and this result coincide with the one which given in [36].

Now let make a simulation of the first integral (VI.6.53) on $\mathbb{T}$ as defined above and let the initial condition $q(0) = 1, p(0) = 0$.

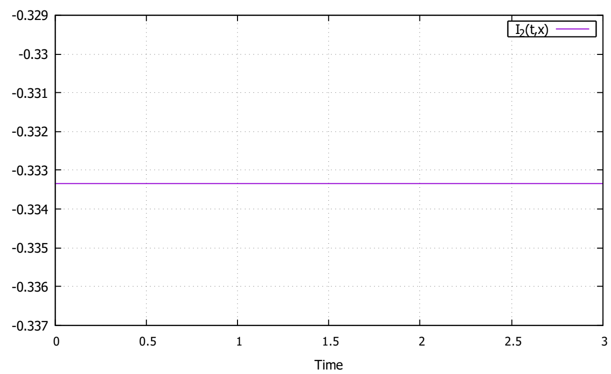


Figure VI.4: Simulation of the quantity (VI.6.53) on time scales $\mathbb{T}$

VI.6.4 Examples of C. J. Song & Y. Zhang

In this section we present two examples which considered in [87].

First example. Taking the following Lagrangian

$$L\left(t, q^\rho, \nabla q\right) = t^2\left[\frac{1}{2}(\nabla q)^2 - \frac{1}{6}(q^\rho)^6\right].$$

The corresponding Hamiltonian function is given by

$$H\left(t, q^\rho, p\right) = \frac{1}{2}\frac{p^2}{t^2} + \frac{1}{6}t^2\left(q^\rho\right)^6 \tag{VI.6.54}$$

Then, the Hamiltonian system is given by

$$\begin{cases} \nabla q = \dfrac{p}{t^2}, \\ \nabla p = -t^2\left(q^\rho\right)^5. \end{cases} \tag{VI.6.55}$$

The authors applied the formula [87, Theorem 4]

$$I^{SZ}(q,p) = p(t)\cdot\xi(t,q(t),p(t)) - \left(H(t,q^{\rho}(t),p(t)) + \nu(t)\,\frac{\partial H}{\partial t}(t,q^{\rho}(t),p(t))\right)\xi_0$$

to this example with the infinitesimals $\xi_0 = -2t$, $\xi = q$, they get the following conserved quantity:

$$I_1^{SZ}(q,p) = p(t)\cdot q(t) - \frac{(p(t))^2}{3t} + \frac{7}{9}t^3\,(q^{\rho}(t))^6\,.$$

Second example. Taking the following Lagrangian

$$L\left(t,q_1^{\sigma},q_2^{\sigma},\Delta q_1,\Delta q_2\right) = \frac{1}{2}\left[(\Delta q_1)^2 + (\Delta q_2)^2\right] - q_2^{\sigma},$$

The corresponding Hamiltonian is given by

$$H\left(t,q_1^{\sigma},q_2^{\sigma},p_1,p_2\right) = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_2^{\sigma}. \tag{VI.6.56}$$

Thus, the Hamiltonian system is given by

$$\begin{cases} \Delta q_1 = p_1, \\ \Delta q_2 = p_2, \\ \Delta p_1 = 0, \\ \Delta p_2 = -1. \end{cases} \tag{VI.6.57}$$

The authors applied the following formula [87, Theorem 7]

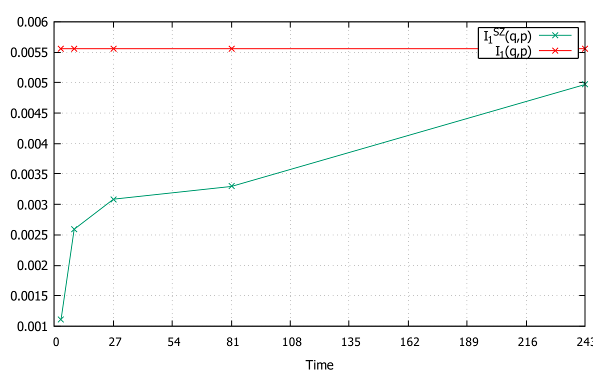
$$I^{SZ}(q,p) = p(t)\cdot\xi(t,q(t),p(t)) + \left[\mu(t)\,\frac{\partial H}{\partial t}\left(t,q^{\sigma}(t),p(t)\right) - H\left(t,q^{\sigma}(t),p(t)\right)\right]\xi_0(t,q(t),p(t))$$

to this example with the infinitesimals $\xi_0 = p_1$, $\quad\xi_1 = \xi_2 = 0$, they assert that the quantity

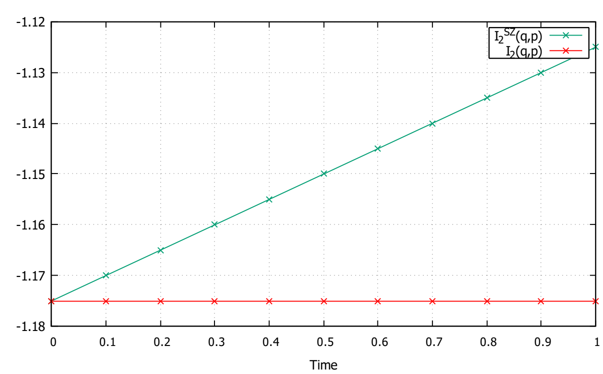
$$I_2^{SZ}(q,p) = p_1(t)\left[\frac{1}{2}\left(p_1^2(t) + p_2^2(t)\right) + q_2^{\sigma}(t)\right]$$

is a first integral.

Numerical test. Let the time scales $\mathbb{T}_1 = \{3^n,\ n \in \mathbb{N}\cup\{0\}\}$ and $\mathbb{T}_2 = \{hk, \quad k \in \mathbb{Z}\}$ with $h > 0$ a numerical test of the quantities I_1^{SZ} and I_2^{SZ} are presented in the following. The initial conditions for I_1^{SZ}, we choose $q(1) = 0$, $p(1) = 0.1$ on $\mathbb{T}_1$ with $n = 5$. For I_2^{SZ}, we choose $q_1(0) = 0.8$, $p_1(0) = 1$, $q_2(0) = p_2(0) = 0.5$ on $\mathbb{T}_2$ with $h = 0.1$.



(a) The quantity I_1^{SZ} on time scales $\mathbb{T}_1$



(b) The quantity I_2^{SZ} on time scales $\mathbb{T}_2$

Observing these figures, it can asset that both of the quantities I_1^{SZ} and I_2^{SZ} are not constant of motion. So that these results are wrong.

Part C

Continuous and Discrete Eringen's Nonlocal Elastica

Chapter VII

Integrating Factor for Eringen's Nonlocal Elastica

In this chapter, we prove by using Helmholtz's conditions that Eringen's nonlocal elastica [23] defined by

$$a(x)\ddot{x} + b(x)(\dot{x})^2 + c(x) = 0 \quad \textit{with} \quad b = a', \quad c = \frac{1}{l^2}a'$$

does not possess a Lagrangian formulation. We find that the function $\psi(x) = a(x)$ is a variational integrating factor and the Lagrangian associated to this equation given by

$$L(x, v) = \frac{1}{2}a^2v^2 + \frac{1}{2l^2}(a')^2 + \frac{1}{l^2}a''.$$

This Chapter is based on section 2 and 3 of the accepted article "About the structure of the discrete and continuous Eringen's nonlocal elastica" *with J. Cresson, Mathematics and Mechanics of Solids, 2022, in Press.*

VII.1 Introduction

The Euler-Lagrangian equation is a specific class of second order differential equations, that is arising from a variational principle. However, from the reverse viewpoint, for a given differential equation, one needs to find out whether is variational or not. The inverse problem refers to as the Helmholtz's inverse problem of the calculus of variations [50]. The abstract Helmholtz's theorem is formulated using the concept of self-adjointness of Fréchet derivative for a differential operator associated to a given differential equation. For a deeper discussion, we refer the reader to ([9], [41], [83]).

In two letters to one of the author [21], N. Challamel raised a number of issues concerning the *continuous Eringen's nonlocal elastica equation* defined by

$$\left(1 - \beta l^2 \cos(x)\right)\ddot{x} + \beta\left(1 + l^2\dot{x}^2\right)\sin(x) = 0, \qquad \text{with} \qquad \dot{x}(0) = \dot{x}(1) = 0, \qquad \text{(VII.1.1)}$$

where $\beta \in \mathbb{R}$, $l^2 \in [0,1]$ and where $\dot{x}$ (resp. $\ddot{x}$) denotes the first (resp. second) derivative of x with respect to t and its *discrete analogue* defined by N. Challamel and al. in [22] by

$$x_{i+1} - 2x_i + x_{i-1} = -\frac{\beta}{n^2}\sin(x_i), \qquad \text{(VII.1.2)}$$

with the boundary conditions $x_1 = x_0$ and $x_{n-1} = x_n$, using the continualization method as exposed for example in [22].

Putting apart the boundary conditions, we are interested in the algebro-geometric structure of these two dynamical systems and their relation.

The main point is that the Eringen's nonlocal elastica does not possess a specific geometrical or algebraic structure which can be used to constraint the discrete analogue one is looking for. This equation does not possess a Lagrangian formulation. The aim is to construct a variational integrating factor, i.e., a function ψ such that the equation multiplied by ψ possesses a Lagrangian formulation.

Organization of the chapter. Section VII.2 contains reminders about abstract Helmholtz's conditions. In Section VII.3, we introduce a family of ordinary differential equations called the *Eringen's family* generalizing the classical Eringen's nonlocal elastica for which we explicit the *necessary and sufficient Helmholt'z conditions* for the existence of a Lagrangian variational formulation (see [83]). In particular, we prove that the Eringen's nonlocal elastica does not possess a variational formulation giving a formal proof of arguments and statements given by N. Challamel and al. in [23]. In Section VII.4, we characterize the subfamily of the Eringen's family for which an integrating exists, i.e., a function such that the given equation multiplied by this function possesses a Lagrangian variational formulation. In particular, we are able to provide an *explicit integrating factor* for the Eringen's nonlocal elastica.

VII.2 Reminder about abstract Helmholtz's conditions

Let O be a second order differential operator,

$$\begin{array}{rccc} O: & \mathscr{C}^2([a,b],\mathbb{R}) & \longrightarrow & \mathscr{C}^2([a,b],\mathbb{R}) \\ & x & \longmapsto & O[x]. \end{array}$$

Fréchet derivative. Recall that the Fréchet derivative associated to O along the direction h defined by

$$DO[x](w) = \lim_{\epsilon \to 0} \frac{O[x+\epsilon w] - O[x]}{\epsilon}. \qquad \text{(VII.2.1)}$$

Adjoint operator. The adjoint of the operator DO, denoted by $\big(DO[x]\big)^*$, can be determined for all $(w,z) \in \mathscr{C}^2([a,b],\mathbb{R}) \times \mathscr{C}_c^2([a,b],\mathbb{R})$ by

$$\int_a^b DO[x](w)\, z\; dt = \int_a^b w\, \big(DO[x]\big)^*(z)\; dt,$$

Self-adjointness. The differential operator DO is self-adjoint if and only if $DO[x] = (DO[x])^*$.

We have a sufficient and necessary condition for a differential equation to possess a variational principle.

Theorem VII.1 (Abstract Helmholtz's theorem)**.** *Let O be a differential operator associated to a differential equation $O[x] = 0$. Then, $O[x] = 0$ possess a Lagrangian formulation if and only if the Fréchet derivative associated to O is self-adjoint.*

A proof can be found in the book of Olver ([83, p.377-379] for a historical about the Helmholtz's problem of the calculus of variations and [83, Theorem 5.92 p.364].

VII.3 Explicit Helmholtz's conditions for Eringen's family

We denote by $\mathcal{E}$ the three parameter family of second order differential equations defined by

$$O[x] = a(x)\ddot{x} + b(x)(\dot{x})^2 + c(x) = 0, \tag{VII.3.1}$$

where a, b, c are real functions. We call **Eringen's family** the previous set of second order differential equations.

This terminology is suggested by the fact that the Eringen's nonlocal elastica belongs to $\mathcal{E}$ with

$$a(x) = 1 - \beta l^2 \cos x, \quad b = a', \quad c = \frac{1}{l^2}a'. \tag{VII.3.2}$$

An element of $\mathcal{E}$ is denoted by $\mathcal{E}_{a,b,c}$.

A natural question is to characterize the sub-family of equations which are Lagrangian. This can be done using the **Helmholtz's criterion**: Let O be the differential operator associated to (VII.3.1) and given by

$$O = a(\cdot)\frac{d^2}{dt^2} + b(\cdot)\left(\frac{d}{dt}\right)^2 + c(\cdot) \tag{VII.3.3}$$

Applying the result in Theorem VII.1, one can obtain explicit conditions ensuring that (VII.3.1) is Lagrangian.

Lemma VII.1. *The second order differential equation* (VII.3.1) *is Lagrangian if and only if $a' = 2b$.*

Proof. The Fréchet derivative of the differential operator is given by

$$DO[x](w) = a\ddot{w} + a'\ddot{x}w + 2b\dot{x}\dot{w} + b'(\dot{x})^2 h + c'w. \tag{VII.3.4}$$

The adjoint is then given by

$$\begin{aligned}\big(DO[x]\big)^{\star}(w) &= \frac{d^2}{dt^2}(aw) + a'\ddot{x}w - \frac{d}{dt}(2b\dot{x}w) + b'(\dot{x})^2 w + c'w, \\ &= \ddot{a}w + 2\dot{a}\dot{w} + a\ddot{w} + a'\ddot{x}w - 2\dot{b}\dot{x}w - 2b\dot{x}\dot{w} - 2b\ddot{x}w + b'(\dot{x})^2 w + c'w.\end{aligned} \tag{VII.3.5}$$

Using the fact that for an arbitrary function f, we have $\dot{f}(x) = \dot{x}f'(x)$ and $\ddot{f}(x) = \ddot{x}f'(x) + (\dot{x})^2 f''(x)$, we obtain

$$\begin{aligned}\big(DO[x]\big)^{\star}(w) &= \left(\ddot{x}a' + (\dot{x})^2 a''\right)w + 2\dot{x}a'\dot{w} + a\ddot{w} + a'\ddot{x}w - 2b'(\dot{x})^2 w - 2b\ddot{x}w - 2b\dot{x}\dot{w} + b'(\dot{x})^2 w + c'w, \\ &= a\ddot{w} + \left(2\dot{x}a' - 2b\dot{x}\right)\dot{w} + \Big(\ddot{x}a' + (\dot{x})^2 a'' + a'\ddot{x} - 2b'(\dot{x})^2 - 2b\ddot{x} + b'(\dot{x})^2 + c'\Big)w\end{aligned} \tag{VII.3.6}$$

The self adjoint property gives the following set of relations

$$\begin{cases} 2\left(a' - b\right)\dot{x} &= 2b\dot{x}, \\ \ddot{x}a' + (\dot{x})^2 a'' + a'\ddot{x} - 2b'(\dot{x})^2 - 2b\ddot{x} + b'(\dot{x})^2 + c' &= a'\ddot{x} + b'(\dot{x})^2 + c', \end{cases} \tag{VII.3.7}$$

which leads to

$$\begin{cases} a' - b & = b, \\ \ddot{x}\left(a' - 2b\right) + (\dot{x})^2\left(a'' - 2b'\right) & = 0, \end{cases} \tag{VII.3.8}$$

Of course, the first equation

$$a' - 2b = 0, \tag{VII.3.9}$$

implies the second one. This concludes the proof. □

Applying this result to the Eringen's nonlocal elastica $\mathcal{E}_{a,b,c}$ we deduce that:

Lemma VII.2. *The Eringen's nonlocal elastica equation does not possess a Lagrangian formulation.*

Proof. As $b = a'$ the Helmholtz's condition reduces to $a' = 0$ which is not true. □

This result has been stated in [23, p.132]. The previous result can be considered as a complete proof of this statement.

VII.4 Integrating factor and the Helmholtz's conditions

The Helmholtz's conditions are deeply related to the presentation of the equation. In particular, even if $O[x] = 0$ does not satisfy the conditions, the equation $\psi O[x] = 0$ with ψ a suitable function of x which is not zero almost everywhere, although equivalent to the initial equation can possess a Lagrangian formulation. The function ψ is then called an *integrating factor*.

Using Lemma VII.1, we have the following characterization of admissible integrating factors:

Lemma VII.3. *Let us consider a differential equation of the form* (VII.3.1) *such that* $a' - 2b = f$ *with* $f \neq 0$. *Let* ψ *be an almost everywhere non zero function. Then the differential equation* (VII.3.1) *admits* ψ *as an integrating factor if*

$$\psi' a + \psi f = 0. \tag{VII.4.1}$$

Applying this result on the Eringen's nonlocal elastica, we obtain the following result:

Theorem VII.2. *The Eringen's nonlocal elastica possesses* $\mathcal{E}_{a,b,c}$ *a unique (up to multiplication by a constant) integrating factor given by the function* a. *A possible Lagrangian is given by*

$$L(x, v) = \frac{1}{2}a^2v^2 + \frac{1}{2l^2}(a')^2 + \frac{1}{l^2}a''. \tag{VII.4.2}$$

Proof. In the Eringen's nonlocal elastica case, we have $f = -a'$ so that (VII.4.1) is equivalent to

$$\psi' a - \psi a' = 0. \tag{VII.4.3}$$

As a is almost everywhere non zero, this equation can be solved explicitly ans gives

$$\psi = Ca, \tag{VII.4.4}$$

where C is a constant. As a consequence, ψ is an admissible function and the a-deformation of the Eringen's nonlocal elastica equation possesses a Lagrangian formulation.

Using the proposed Lagrangian, we obtain

$$\frac{\partial L}{\partial v} = a^2 v, \quad \frac{\partial L}{\partial x} = a' a v^2 + \frac{1}{l^2} a' a'' + \frac{1}{l^2} a^{(3)}, \tag{VII.4.5}$$

where $a^{(3)}$ denotes the third derivative of a with respect to x. Using the fact that

$$a^{(3)} = -a', \tag{VII.4.6}$$

we obtain for the Euler-Lagrange equation

$$\frac{d}{dt}\left(a^2 \dot{x}\right) = a' a \left(\dot{x}\right)^2 + \frac{1}{l^2} a' a'' - \frac{1}{l^2} a', \tag{VII.4.7}$$

which gives using the fact that $a'' = 1 - a$ that

$$\begin{aligned} \frac{d}{dt}\left(a^2 \dot{x}\right) &= a' a \left(\dot{x}\right)^2 - \frac{1}{l^2} a' a \\ &= a \left(a' \left(\dot{x}\right)^2 - \frac{1}{l^2} a'\right). \end{aligned} \tag{VII.4.8}$$

As

$$\frac{d}{dt}\left(a^2 \dot{x}\right) = 2 \left(\dot{x}\right)^2 a' a + a^2 \ddot{x} = a \left(2 \left(\dot{x}\right)^2 a' + a \ddot{x}\right), \tag{VII.4.9}$$

we obtain, introducing this expression in equation (VII.4.8) that

$$a \left(\left(\dot{x}\right)^2 a' + a \ddot{x} + \frac{1}{l^2} a'\right) = 0, \tag{VII.4.10}$$

which concludes the proof. □

Chapter VIII

Variational structure for the Continuous Eringen's Nonlocal Elastica

In this chapter, we exploit the result of Chapter VII to derive a Hamiltonian structure associated to the Eringen's nonlocal elastica. Explicit expressions of the solutions in term of elliptic integrals of the first kind are then deduced.

*This Chapter is based on section 4-6 of the accepted article "*About the structure of the discrete and continuous Eringen's nonlocal elastica*" with J. Cresson, Mathematics and Mechanics of Solids, 2022, in Press.*

VIII.1 Introduction

As we have seen in Chapter VII, the Eringen's nonlocal elastica does not possess a a Lagrangian formulation. By means of *variational integrating factor*, such modified equation becomes Lagrangian which enable us to derive a Hamiltonian function and to exhibit an *explicit first integral* for the Eringen's nonlocal elastica.

In [21], N. Challamel suggested that one can probably obtain *explicit formula for the solutions* of the Eringen's nonlocal elastica using *elliptic integrals*. In this chapter, by taking the benefit of the Hamiltonian structure, we prove that this is indeed the case using elliptic integrals of the first kind and simplifying previous result of M. Lembo [66], [65], [64].

Organization of the chapter. In Section VIII.2, we derive the Hamiltonian associated to the modified Eringen's nonlocal elastica. We deduce an *explicit first integral*. The first integral is then used to provide *explicit formula for the solutions* in term of *elliptic integrals* in Section VIII.4.

VIII.2 A Hamiltonian associated to the modified Eringen's nonlocal elastica equation

The classical way of constructing a Hamiltonian formulation associated to the Lagrangian one via the Legendre transform [4] gives the following result:

Theorem VIII.1. *The Hamiltonian system corresponding to the Eringen's equation is given by*

$$\begin{cases} \dot{x} &= \dfrac{\partial H}{\partial p} = \dfrac{p}{a^2}, \\ \dot{p} &= -\dfrac{\partial H}{\partial x} = a\left(a'\dfrac{p^2}{a^4} - \dfrac{1}{l^2}a'\right), \end{cases} \tag{VIII.2.1}$$

with the Hamiltonian function

$$H(x,p) = \frac{1}{2a^2}p^2 - \frac{1}{2l^2}(a')^2 - \frac{1}{l^2}a''. \tag{VIII.2.2}$$

Proof. The variable p corresponding to the momentum is defined by

$$p = \frac{\partial L}{\partial v} = a^2 v, \tag{VIII.2.3}$$

which is one to one as long as $a \neq 0$.

The Legendre transform gives for the Lagrangian L given in Lemma VII.2 the following Hamiltonian:

$$\begin{aligned} H(x,p) &= pv - L(x,v), \\ &= \frac{p^2}{a^2} - \frac{1}{2}a^2\frac{p^2}{a^4} - \frac{1}{2l^2}(a')^2 - \frac{1}{l^2}a'', \\ &= \frac{1}{2a^2}p^2 - \frac{1}{2l^2}(a')^2 - \frac{1}{l^2}a''. \end{aligned} \tag{VIII.2.4}$$

A simple computation gives the equation of motion. This concludes the proof. □

It must be noted that the Hamiltonian function depends on the parameters β and l and must be understood as

$$H_{\beta,l}(x,p) = \frac{1}{2a_{\beta,l}^2}p^2 - \frac{1}{2l^2}(a'_{\beta,l})^2 - \frac{1}{l^2}a''_{\beta,l}, \tag{VIII.2.5}$$

with

$$a_{\beta,l}(x) = 1 - \beta l^2\cos(x). \tag{VIII.2.6}$$

As we have

$$a'_{\beta,l}(x) = \beta l^2 \sin(x), \quad \text{and} \quad a''_{\beta,l}(x) = \beta l^2\cos(x), \tag{VIII.2.7}$$

we have explicitly the Hamiltonian

$$H_{\beta,l}(x,p) = \frac{1}{2(1-\beta l^2\cos(x))^2}p^2 - \frac{1}{2}\beta^2 l^2(\sin(x))^2 - \beta\cos(x). \tag{VIII.2.8}$$

Taking $l = 0$ in the previous Hamiltonian, we obtain the classical **simple pendulum** equation:

Corollary VIII.1. *Let $l = 0$, then the Hamiltonian system Eqs. (VIII.2.1) corresponds to the simple pendulum motion:*

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -\beta\sin(x). \end{cases} \tag{VIII.2.9}$$

with the Hamiltonian

$$H_{\beta,0}(x,p) = \frac{p^2}{2} - \beta\cos(x). \tag{VIII.2.10}$$

This property can be used to deduce interesting **qualitative properties** if the Eringen's nonlocal elastica using perturbation theory and the fact that the Hamiltonian is a **constant of motion** on the solutions of the system.

The shape of the energy manifold looks as follows:

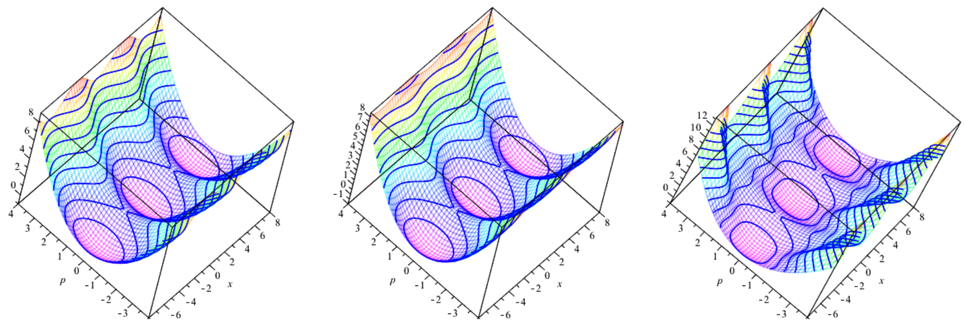


Figure VIII.1: Energy manifold for $\beta = 1$ and $l = 0$, $l = 0.2$ and $l = 0.5$

VIII.3 Qualitative behavior of the Eringen's nonlocal elastica solutions

As already reminded in Section VIII.2, the main consequence of the existence of a Hamiltonian structure given by Theorem VIII.1 is the fact that it provides a constant of motion, i.e., that for all solutions $(x(t)$,$p(t))$ of the Hamiltonian equation (VIII.2.1), we have

$$H_{\beta,l}(x(t), p(t)) = H_{\beta,l}(x(0), p(0)), \tag{VIII.3.1}$$

for all $t \in \mathbb{R}$.

Giving the Legendre transform, it means that we have the following result:

Lemma VIII.1. *Let x be a solution of the Eringen's nonlocal elastica equation, then the function $H(x, a^2\dot{x})$ is constant.*

Proof. This follows directly from the fact that for any solution (x_t, p_t) of the Hamiltonian system, we have $H(x_t, p_t)$ which is a constant function. As $p_t = a^2\dot{x}$ by the Legendre transform and x_t is by construction a solution of the Eringen's nonlocal elastica, we obtain the result. □

A natural idea is then to look at the **level sets** of the function $I_{\beta,l} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as

$$I_{\beta,l}(x, v) = H_{\beta,l}(x, a^2(x)v), \tag{VIII.3.2}$$

in order to have a global view of the qualitative behavior of the solutions of the Eringen's nonlocal elastica.

In the following, we provide the level set of $H_{1,l}$ and $I_{1,l}$ for different values of l and we compare with the phase portrait of the Eringen's nonlocal elastica showing the strong influence of the first integral on the dynamics.

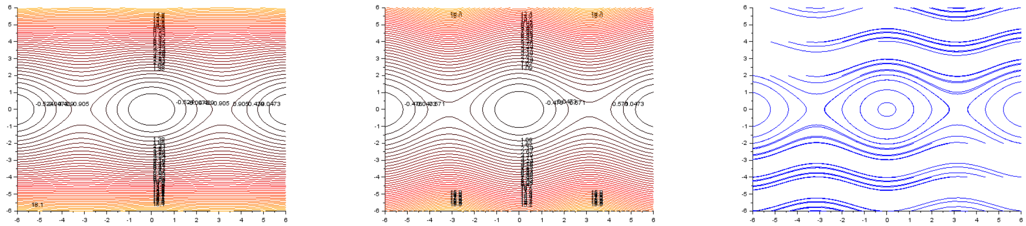


Figure VIII.2: Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.2)$

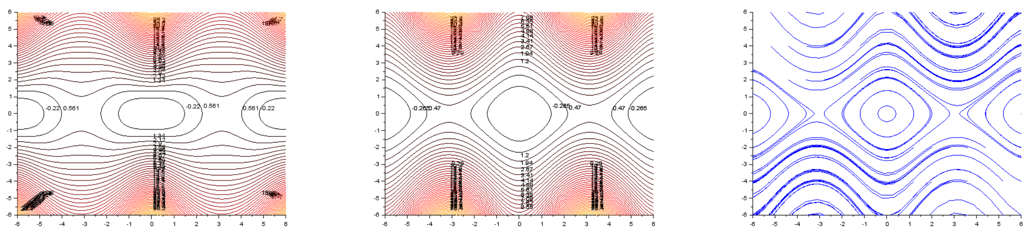


Figure VIII.3: Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.5)$

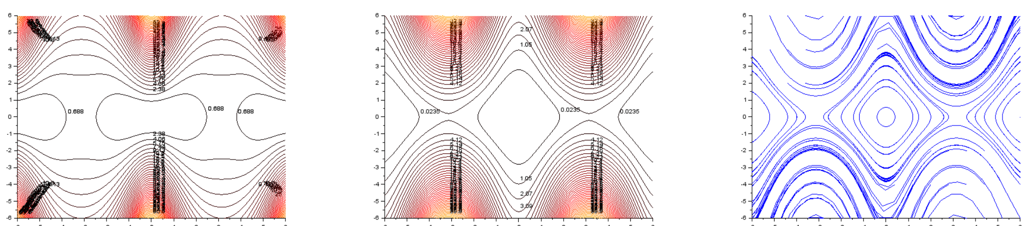


Figure VIII.4: Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.7)$

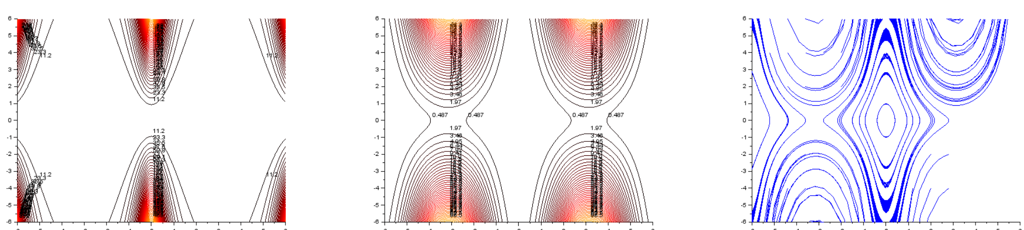


Figure VIII.5: Level sets of $H_{\beta,l}$, $I_{\beta,l}$ and phase portrait of the Eringen's nonlocal elastica for $(\beta, l) = (1, 0.9)$

Another representation can be obtained taking into account the 2π-periodicity of the solutions of the equations. In that case, the phase portrait must be given on $\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}$, i.e., $S^1\times\mathbb{R}$ where S^1 is the unit circle, i.e., a cylinder, which is the classical phase space of the simple pendulum equation.

Figure VIII.6: Phase portrait of the Eringen's nonlocal elastica for $\beta = 1$, $l = 0$, $l = 0.2$, $l = 0.5$, $l = 0.7$ and $l = 0.9$

VIII.4 Explicit computation of the solutions of the Eringen's nonlocal elastica equation

An important problem is to obtain **explicit forms** for the solutions of the Eringen's nonlocal elastica equation. In [21], N. Challamel suggests that one can probably obtain such solutions using **elliptic integrals**.

Taking benefit of the Hamiltonian structure of the equation and as a consequence, of the existence of a constant of motion, we derive explicit expression for the solutions of the Eringen's nonlocal elastica equation. They are indeed expressed using **elliptic integrals of the first kind** as suggested by N. Challamel and the fact that for $l = 0$ the system reduces to the simple pendulum equation.

Moreover, we derive similar expressions using the notion of **canonical coordinates** discussed for example in [56, p.22] taking benefit of the fact that the Eringen's nonlocal elastica is invariant under the time translation symmetry. This is formally equivalent to using the Hamiltonian first integral but the concept can be used for other symmetry groups. We then illustrate this procedure in our case directly on the equation and also on the Lagrangian formulation.

Previous results in this direction have been obtained by M. Lembo, see [65], [66]. The expression of the quantities are different and also the procedure used to derive the explicit form of the solutions does not seem to follow a general scheme which can be applied to other equations.

VIII.4.1 Eringen's solutions via Hamiltonian function

The general case. Let us consider the Hamiltonian $H_{\beta,l}$. The result can be resumed as follows:

Lemma VIII.2. *Let $(x_0, v_0) \in \mathbb{R}^2$ be given. We denote by c_0 the quantity $H_{\beta,l}(x_0, a^2(x_0)v_0) = c_0$ and $\beta l^2 = \lambda$, $2c_0 + \beta^2 l^2 = k_0$, $\beta^2 l^2 = \gamma$ then for all $t \in \mathbb{R}$, the solution $x(t)$ satisfies*

$$t = \mathrm{E}_{\beta,l}(x) := \pm \int_{\cos(x_0)}^{\cos(x)} \frac{\lambda u - 1}{\sqrt{k_0 + 2\beta u - \gamma u^2}} \frac{du}{\sqrt{1-u^2}}.$$

This kind of integral is called an elliptic integral.

Proof. Solving the equation $H_{\beta,l}(x,p) = c$ with respect to p, we have

$$p = \pm\frac{a}{l}\sqrt{2cl^2 + a'^2 + 2a''}$$

From the Hamiltonian system (VIII.2.1), the substitution of $p = a^2\dot{x}$ into the last equation gives

$$\dot{x} = \pm\frac{1}{la}\sqrt{2cl^2 + a'^2 + 2a''} \Rightarrow \frac{dt}{dx} = \pm\frac{la}{\sqrt{2cl^2 + a'^2 + 2a''}}$$

We have explicitly $a(x) = 1-\beta l^2\cos(x),\ a'(x) = \beta l^2\sin(x)$ and $a''(x) = \beta l^2\sin(x)$ and by integrating the right hand side of the last implication we obtain

$$t = \int_{x_0}^{x} \pm\frac{(1-\lambda\cos(z))}{\sqrt{k_0 + 2\beta\cos(z) - \gamma(\cos(z))^2}}dz, \tag{VIII.4.1}$$

where we use the equality $(\sin(x))^2 = 1 - (\cos(x))^2$ and we complete the proof by setting $u = \cos z$. ☐

As you can see, we have no hope to obtain a simpler form in the general case than one using elliptic integrals and the simplest one can be made when $l = 0$ that is the simple pendulum case.

The simple case. Let $l = 0$ in (VIII.4.1), then one can recover the classical case of the simple pendulum equation, that is

$$t = \mathrm{E}_{\beta,0}(x) := \pm\frac{1}{\sqrt{2(c+\beta)}}\int_{x_0}^{x}\frac{dz}{\sqrt{1-k\sin^2(z/2)}}. \tag{VIII.4.2}$$

where $k = \frac{2\beta}{c+\beta}$. The right-hand side of Eq. (VIII.4.2) is an **incomplete elliptic integral of the first kind**.

It is possible to derive explicit expression for the solutions of the Eringen's nonlocal elastica equation from the original equation and also from the corresponding Lagrangian by considering a suitable change of variables the so-called canonical variables (see [56]), and the results are coincided with each other. This is done in the following Section.

VIII.4.2 Eringen's solutions via canonical variables

We follow closely the computations made by P. Hydon in [56, p.22] in order to define canonical coordinates allowing to solve explicitly the equation.

Canonical variables for the original Eringen's equation. Let $l \neq 0$. Consider the Eringen's equation

$$a(x)\ddot{x} + b(x)\dot{x}^2 + kb(x) = 0. \tag{VIII.4.3}$$

where $a(x) = 1 - \beta l^2\cos(x),\ b(x) = a'(x)$ and $k = \frac{1}{l^2}$.

Let us consider a change of variables $(t,x) = (r,w)$ satisfying the following set of constraints:

$$\dot{x} = \dot{w} = \frac{1}{\dot{r}}, \quad \ddot{x} = -\frac{\ddot{r}}{\dot{r}^3}.$$

Rewriting equation (VIII.4.3) in terms of r and w, we obtain

$$-a(w)\frac{\ddot{r}}{\dot{r}^3} + b(w)\frac{1}{\dot{r}^2} + kb(w) = 0 \Rightarrow -a(w)\ddot{r} + b(w)\dot{r} + kb(w)\dot{r}^3 = 0.$$

Setting $z = \dot{r}$, the last equation becomes a *separable first-order ODE* given by

$$\dot{z} = \frac{b(w)}{a(w)}\left(z + kz^3\right),$$

whose solution is given by

$$\frac{z}{\sqrt{1+kz^2}} = ca(w),$$

where $c > 0$ is a constant.
Solving the last equation for z yields,

$$z = \dot{r} = \pm\frac{ca(w)}{\sqrt{1-kc^2a^2(w)}}.$$

Finally, returning to the original variables we have that $\dot{r} = dt/dx$ and integrating the last equation gives

$$t = \pm\int \frac{ca(x)}{\sqrt{1-kc^2a^2(x)}}dx + c_1$$

which is an elliptic integral.

Canonical variables with the Lagrangian L. We can also find the solutions of Eringen's equation using the corresponding Lagrangian L as given in Section VII.4:

$$L(x,v) = \frac{1}{2}a^2v^2 + \frac{1}{2l^2}a'^2 + \frac{1}{l^2}a''.$$

It is obvious that L is invariant under the group of translation in time associated to the infinitesimal generator $\boldsymbol{X} = \frac{\partial}{\partial t}$. Such a symmetry is called a variational symmetry [83]. The canonical variables corresponding to the operator $\boldsymbol{X}$ are given by $(t,x) = (r,w)$, see [56, page 65]. Under such variables one can produce a new Lagrangian $\tilde{L}$ in term of r and w.

We have

$$v = \dot{x} = \dot{w} = \frac{1}{\dot{r}}.$$

Defining the following Lagrangian,

$$\begin{aligned}\tilde{L}(w,\dot{r}) = \frac{1}{\dot{w}}L(x,v) = \frac{1}{v}L(x,v) &= \frac{1}{2}a^2v + \frac{1}{2l^2v}a'^2 + \frac{1}{l^2v}a'' \\ &= \frac{1}{2\dot{r}}a^2(w) + \frac{1}{2l^2}(a'(w))^2\dot{r} + \frac{1}{l^2}a''(w)\dot{r}.\end{aligned}$$

So that, the corresponding Euler Lagrange equation is given by

$$\frac{d}{dw}\left(\frac{\partial\tilde{L}}{\partial\dot{r}}\right) = \frac{\partial\tilde{L}}{\partial r}$$

As $\tilde{L}$ does not depend on r, we have the following first integral:

$$I(w,\dot{r}) = \frac{\partial \tilde{L}}{\partial \dot{r}} = -\frac{1}{2\dot{r}^2}a^2(w) + \frac{1}{2l^2}(a'(w))^2 + \frac{1}{l^2}a''(w).$$

Rewriting I in term of the original variables becomes

$$I(x,v) = -\frac{1}{2}a^2\dot{x}^2 + \frac{1}{2l^2}a'^2 + \frac{1}{l^2}a''.$$

Since $I(x,v)$ is a constant of motion one can write $I(x,v) = c$, where c is a constant. We then obtain

$$\dot{x} = \pm\frac{\sqrt{-2cl^2 + a'^2 + 2a''}}{la} \Longrightarrow t = \pm\int \frac{la}{\sqrt{-2cl^2 + a'^2 + 2a''}}dx + c_1.$$

This kind of integral is also an elliptic one.

Chapter IX

Toward a discrete version of the Eringen's nonlocal elastica

In this chapter, we derive discrete version of the Eringen's nonlocal elastica preserving the Lagrangian and Hamiltonian structure using the result of Chapter VIII, then compare it with Challamel's and co-worker definition of a discrete Eringen's nonlocal elastica.

This Chapter is based on section 7 of the accepted article "About the structure of the discrete and continuous Eringen's nonlocal elastica" *with J. Cresson, Mathematics and Mechanics of Solids, 2022, in Press.*

IX.1 Introduction

Deriving a discrete analogue of a continuous differential equation is always a challenge and is not only a question of discretizing the differential equation using classical tools of numerical analysis. Indeed, doing such a discretization destroys in general the basic algebraic, geometric or qualitative properties of the equations and solutions of the continuous model. An example of well defined discrete analogue is provided by the construction of variational integrators for Euler-Lagrange equations. Indeed, in this case, variational integrators are designed in order to preserve the variational structure at the discrete level and as a consequence most of the qualitative properties of the solutions.

As we have seen in Chapter VIII, the modified Eringen's nonlocal elastica is variational due to the variational integrating factor. The aim of this chapter is to provide a discrete analogue of the Eringen's nonlocal elastica by taking benefit of the existence of a Lagrangian and Hamiltonian structure. Firstly, we derive a *variational integrator* for the Eringen's nonlocal elastica, i.e., a numerical integrator preserving the Lagrangian and Hamiltonian structure at the discrete level. A classical property of variational integrators is that they preserve very well energy, i.e., the evaluation of the Hamiltonian on solutions. This property induce a very good preservation of the first integral at the discrete level. Secondly, as the variational integrator is implicit due to the presence of the integrating factor, we obtain an explicit numerical scheme taking into account the value of the first integral. We call this new numerical integrator a *topological integrator*.

Organization of the chapter. Section IX.3 deals with the construction of an efficient numerical scheme for the Eringen's nonlocal elastica. We use the formalism of discrete embedding in order to derive *variational integrators* for the modified Eringen's nonlocal elastica. Variational integrators are in this case implicit. However, a slight modification of the construction lead to an explicit scheme called a *topological integrator*. These two scheme are implemented and compared with the classical Euler scheme as well as the Challamel and al. discrete Eringen's nonlocal elastica defined in [23] here called *Challamel's integrator*. In particular, we prove that variational integrators and the corresponding discrete Hamiltonian versions are more efficient than the other numerical scheme.

IX.2 Using the classical Euler scheme

In this Section, we provide some simulations of the Eringen's nonlocal elastica using an explicit Euler scheme. The quality of the numerical scheme is measured by the quality of the preservation of the first integral at the discrete level. As we have seen in Section VIII.3, the qualitative properties of the solutions are controlled by the level surface of the first integral.

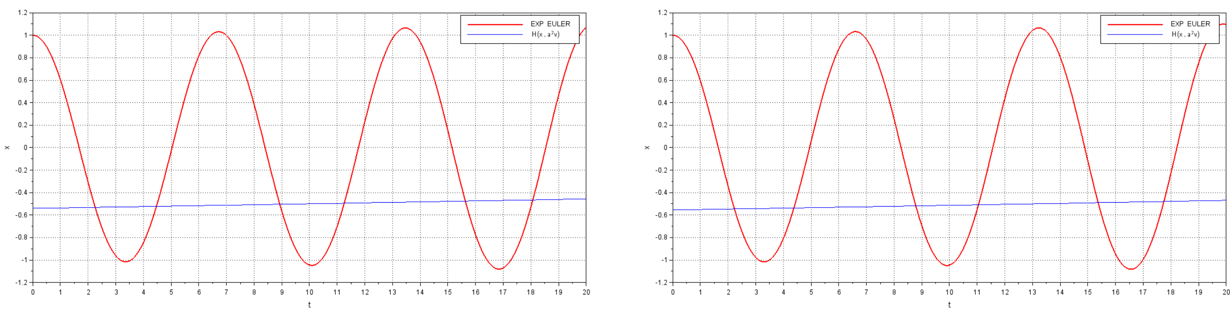


Figure IX.1: Numerical solution of the Eringen's nonlocal elastica for $l = 0$, $l = 0.2$ with the corresponding evaluation of the first integral - Euler scheme - $h = 0.01$

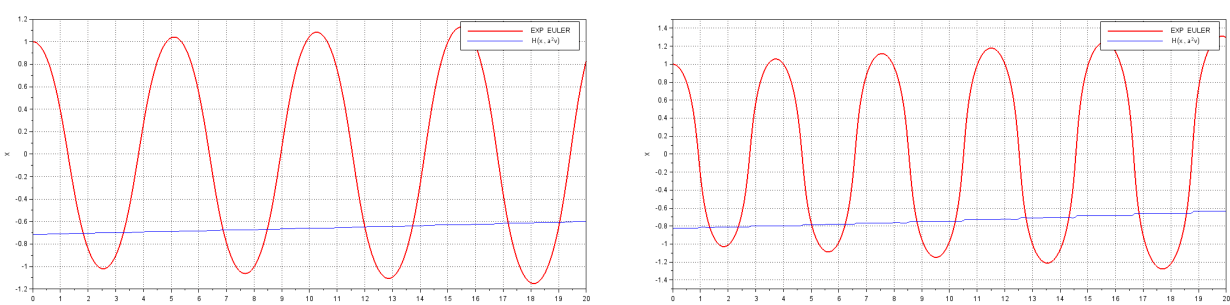


Figure IX.2: Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$, $l = 0.9$ with the corresponding evaluation of the first integral - Euler scheme - $h = 0.01$

As we can see, a rapid divergence of the values of the first integral is observed.

IX.3 Variational and Topological integrators for the Eringen's nonlocal elastica

The existence of the first integral $I_{\beta,l}$ for the Eringen's nonlocal elastica can be used to design numerical integrators preserving this first integral. Such a numerical integrator is reminiscent of geometric numerical integration and can be called topological numerical integrator as the preservation of the first integral ensure that the topological properties of the solutions constrained by the first integral are preserved. To construct such a topological numerical integrator an idea is first to use the existence of an integrating factor and the variational structure which is associated. We then first construct a variational integrator for the modified Eringen's nonlocal elastica using the **discrete embedding formalism** given in Chapter II (see also [29]). Having this numerical integrator, we are able to propose a discrete dynamical systems which can be called "discrete Eringen's nonlocal elastica" as the fundamental properties of this discrete system are similar to the continuous case from the point of view of first integral and existence of an underlying variational structure up to an integrating factor.

IX.3.1 Variational integrator and the Eringen's nonlocal elastica

We begin by recalling the notion of the discrete derivatives and integrals that used in Chapter II. Let $N \in \mathbb{N}^*$, $h = 1/N$ and $\mathbb{T} = \{t_i = a + ih,\ i = 0, \dots, N\}$ be the uniform time scales and let $x \in \mathrm{C}(\mathbb{T}, \mathbb{R})$.

The $\pm$-discrete integrals. Let $t \in \mathbb{T}$. The $\pm$-discrete integrals of x over $[a,t]$ are the quantities defined by

$$\int_{[a,t]_{\mathbb{T}}} x(s)\ \Delta_+ s = h \sum_{t_i \in [a,t] \cap \mathbb{T}^+} x(t_i) \quad \left(\text{resp. } \int_{[a,t]_{\mathbb{T}}} x(s)\ \Delta_- s = h \sum_{t_i \in [a,t] \cap \mathbb{T}^-} x(t_i).\right) \tag{IX.3.1}$$

Of course $\Delta_\pm$-integrals correspond to the right Riemann (resp. left Riemann) sum.

The $\Delta_\pm$- derivatives. Refer to the forward (resp. backward) discrete derivatives defined for all $x \in \mathrm{C}(\mathbb{T}, \mathbb{R})$ by

$$\Delta_+[x](t_i) = \frac{x_{i+1} - x_i}{h},\ i = 0, \dots, N-1 \quad \left(\text{resp. } \Delta_-[x](t_i) = \frac{x_i - x_{i-1}}{h},\ i = 1, \dots, N.\right) \tag{IX.3.2}$$

In the following, we simply denote $\Delta_+[x]_i$ (resp. $\Delta_+[x]_i$) for $\Delta_-[x]_i(t_i)$ (resp. $\Delta_-[x](t_i)$) when no confusion is possible.

The discrete derivatives and integrals satisfy a discrete analogue of the fundamental theorem of differential calculus, i.e.,

$$\Delta_\pm \left[\int_a^t f(s) \Delta_\pm s \right](t) = f(t) \quad \forall\ t \in \mathbb{T}^\pm. \tag{IX.3.3}$$

IX.3.2 Variational integrators and discrete embedding

In this subsection, we follow the discrete embedding formalism used in Chapter II (see Section II.9).

We first define a discrete analogue of the Lagrangian functional

$$\mathcal{L}(x) = \int_a^b L(x(t), \dot{x}(t))\, dt, \tag{IX.3.4}$$

with L given by Theorem VII.2. The $\pm$-discrete Lagrangian functional denoted by $\mathcal{L}^h_\pm$ by

$$\mathcal{L}^h_\pm[x] = \int_a^b L\left(x, \Delta_\pm[x]\right)\Delta_\pm t. \tag{IX.3.5}$$

The discrete Euler-Lagrange equation are defined, respectively, by (see Chapter II, Theorem II.5)

$$\begin{aligned} \Delta_-\left[\frac{\partial L}{\partial v}\left(x, \Delta_+[x]\right)\right](t) &= \frac{\partial L}{\partial x}\left(x, \Delta_+[x]\right)(t),\ \forall\, t \in \mathbb{T}^\pm, \\ \Delta_+\left[\frac{\partial L}{\partial v}\left(x, \Delta_-[x]\right)\right](t) &= \frac{\partial L}{\partial x}\left(x, \Delta_-[x]\right)(t),\ \forall\, t \in \mathbb{T}^\pm. \end{aligned} \tag{IX.3.6}$$

The particular feature of the previous numerical integrator is to provide a symplectic integrator which are known to possess very good properties of preservation of energy, i.e., of H.

IX.3.3 The discrete Eringen's nonlocal elastica

We use the previous construction using the Lagrangian obtained in Theorem VII.2. The backward discrete Lagrangian functional is given

$$\begin{aligned} \mathcal{L}^h_-[x] &= \int_a^b \left(\frac{1}{2}\left(a(x)\Delta_-[x]\right)^2 + \frac{1}{2l^2}(a'(x))^2 + \frac{1}{l^2}a''(x)\right)\Delta_- t \\ &= \sum_{i=1}^N \left(\frac{1}{2}(a_i\left(\Delta_-[x]_i\right)^2 + \frac{1}{2l^2}(a'_i)^2 + \frac{1}{l^2}a''_i\right), \end{aligned} \tag{IX.3.7}$$

where $a_i = a(x_i)$, $a'_i = a'(x_i)$ and $a''_i = a''(x_i)$.

The discrete Euler-Lagrange equation associated to L is given by the following Theorem:

Theorem IX.1. *The backward variational integrator associated to Eq.* (VII.1.1) *is given by*

$$a^2_{i+1}(x_{i+1} - x_i) = a^2_i x_i - a^2_i x_{i-1} + a_i\beta\sin(x_i)\left(l^2(x_i - x_{i-1})^2 - h^2\right) \tag{IX.3.8}$$

for $i = 1, \ldots, N-1$

When $l = 0$ and $\beta = 1$, we obtain the classical variational integrator for the simple pendulum:

$$x_{i+1} = 2x_i - x_{i-1} - h^2\sin(x_i). \tag{IX.3.9}$$

It must be noted that the implicit character of the numerical scheme is directly related to the term corresponding to the integrating factor.

Proof. The equation (IX.3.12) can be found directly from (VII.4.5) using the fact that

$$\frac{1}{l^2}a'(a'' - 1) = -\frac{1}{l^2}aa', \tag{IX.3.10}$$

so that

$$\frac{\partial L}{\partial x} = aa' \left[v^2 - \frac{1}{l^2} \right] \tag{IX.3.11}$$

We then obtain the discrete Euler-Lagrange equation

$$\Delta_+ \left[a^2 \, \Delta_-[x] \right]_i = a_i a_i' \left[(\Delta_-[x])_i^2 - \frac{1}{l^2} \right], \quad i = 1, \dots, n-1. \tag{IX.3.12}$$

Using the expression of Δ_+ and Δ_- a more explicit form can be obtained. We have

$$\begin{aligned} h\Delta_+ \left[a^2 \cdot \Delta_-[x] \right]_i &= a_{i+1}^2 \Delta_-[x]_{i+1} - a_i^2 \Delta_-[x]_i, \\ &= h^{-1} \left(a_{i+1}^2 x_{i+1} - (a_{i+1}^2 + a_i^2) x_i + a_i^2 x_{i-1} \right). \end{aligned} \tag{IX.3.13}$$

The right hand term is given by

$$a_i a_i' \left[(\Delta_-[x])_i^2 - \frac{1}{l^2} \right] = a_i a_i' h^{-2} \left((x_i - x_{i-1})^2 - h^2 l^{-2} \right). \tag{IX.3.14}$$

As a consequence, we obtain the following expression

$$a_{i+1}^2 x_{i+1} - (a_{i+1}^2 + a_i^2) x_i + a_i^2 x_{i-1} = a_i a_i' \left((x_i - x_{i-1})^2 - h^2 l^{-2} \right) \tag{IX.3.15}$$

for $i = 1, \dots, N-1$ which can be rewritten as

$$a_{i+1}^2 (x_{i+1} - x_i) = a_i^2 x_i - a_i^2 x_{i-1} + a_i a_i' \left((x_i - x_{i-1})^2 - h^2 l^{-2} \right) \tag{IX.3.16}$$

for $i = 1, \dots, N-1$. This concludes the proof. □

In the same way, the forward variational integrator is given by:

Theorem IX.2. *The forward variational integrator associated to Eq. (VII.1.1) is given by*

$$x_{i+1} = \frac{1}{a_i^2} \left[(a_i^2 + a_{i-1}^2) x_i - a_{i-1}^2 x_{i-1} + a_i l^2 \sin(x_i)(x_{i+1} - x_i)^2 - h^2 a_i \sin(x_i) \right], \quad i = 1, \dots, N-1. \tag{IX.3.17}$$

for $i = 1, \dots, N-1$

When $\beta = 1$ and $l = 0$, we obtain again the classical variational integrator for the simple pendulum

$$x_{i+1} = 2x_i - x_{i-1} - h^2 \sin(x_i). \tag{IX.3.18}$$

When $l \neq 0$, the numerical scheme is implicit but relies on finding roots of a polynomials of degree 2. Precisely, in order to find x_{i+1}, we have to solve the polynomial equation $P_i(x) = 0$ where the polynomial P_i is given by

$$P_i(x) = \alpha_i x^2 - x\beta_i + \gamma_i, \tag{IX.3.19}$$

with

$$\alpha_i = a_i l^2 \sin(x_i), \quad \beta_i = 2a_i l^2 x_i \sin(x_i) + a_i^2, \quad \gamma_i = \left(a_i^2 + a_{i-1}^2 \right) x_i - a_{i-1}^2 x_{i-1} - h^2 a_i \sin(x_i). \tag{IX.3.20}$$

Proof. We have

$$\begin{aligned}\Delta_-\left[a^2(x)\Delta[x]_i\right] &= \frac{1}{h}\left(a_i^2\Delta[x]_i - a_{i-1}^2\Delta[x]_{i-1}\right),\\ &= \frac{1}{h^2}\left(a_i^2(x_{i+1}-x_i) - a_{i-1}^2(x_i - x_{i-1})\right),\\ &= \frac{1}{h^2}\left(a_i^2 x_{i+1} - (a_i^2 + a_{i-1}^2)x_i + a_{i-1}^2 x_{i-1}\right).\end{aligned} \tag{IX.3.21}$$

Moreover, we have

$$\begin{aligned}\frac{\partial L}{\partial x}(x_i,\Delta[x]_i) &= l^2\sin(x_i)a_i(\Delta[x]_i)^2 - \sin(x_i)a_i,\\ &= a_i\sin(x_i)(l^2(\Delta_+[x]_i)^2 - 1),\end{aligned} \tag{IX.3.22}$$

for $i = 1,\dots,N-1$. This concludes the proof. □

In the forward and backward case the corresponding variational integrators are **implicit**. This little increase of the algorithmic complexity is the price to pay in order to obtain a variational integrator in this case. In Section IX.3.5, we look for a modification of the scheme which can lead to an explicit one.

IX.3.4 Simulations of the variational integrator for the Eriengen's nonlocal elastica

In order to implement the variational integrator for the Eringen's nonlocal elastica, we need to solve the implicit equation. This is done numerically using a Newton-Raphson method.

Simulations of the variational integrator are provided in the following for $h = 0.1$ on the interval $[0, 20]$ with $x_0 = 1$, $x_1 = x_0$ for $l = 0$, 0.2, 0.5, 0.7 and 0.9.

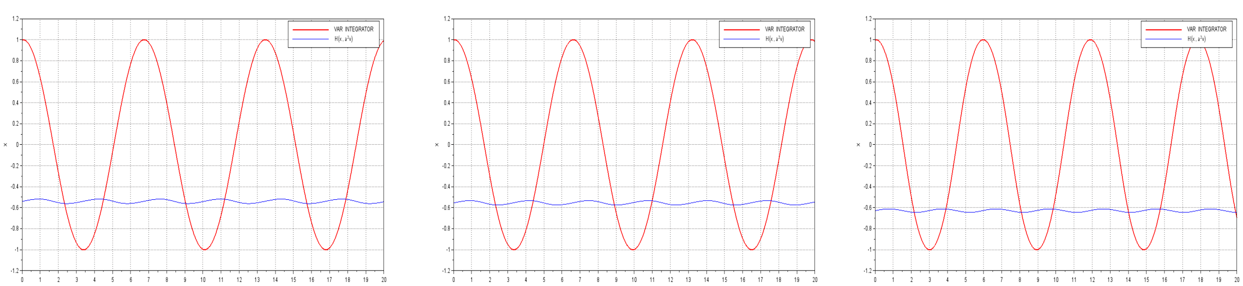


Figure IX.3: Numerical solution of the Eringen's nonlocal elastica for $l = 0$, $l = 0.2$, $l = 0.7$ $l = 0.9$ with the corresponding evaluation of the first integral - Variational integrator - $h = 0.1$

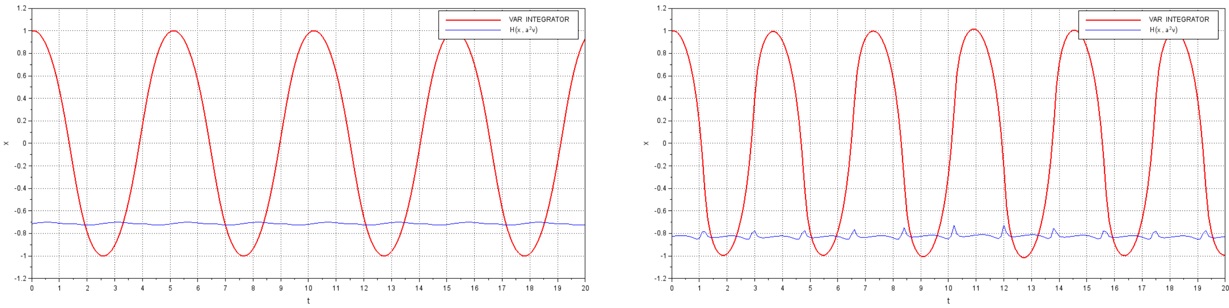


Figure IX.4: Numerical solution of the Eringen's nonlocal elastica for $l = 0$, $l = 0.2$, $l = 0.7$ $l = 0.9$ with the corresponding evaluation of the first integral - Variational integrator - $h = 0.1$

We have then a very good preservation of the first integral and an accurate simulation of the behavior of the solutions.

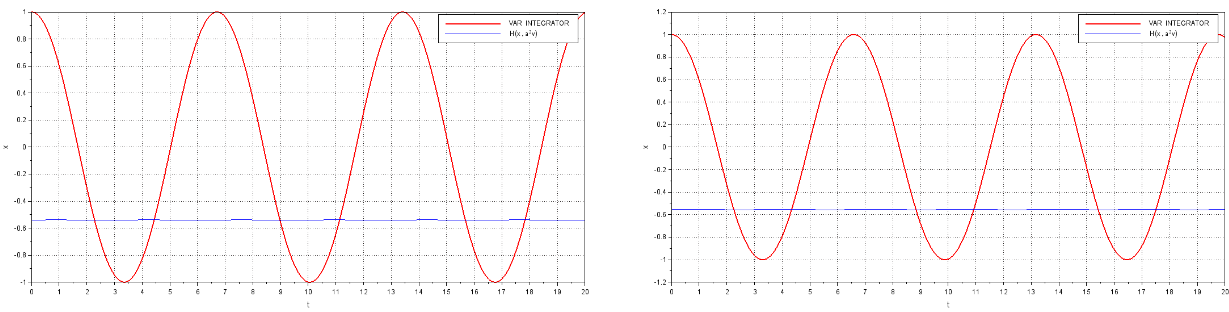


Figure IX.5: Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ with the corresponding evaluation of the first integral - Variational integrator - $h = 0.01$

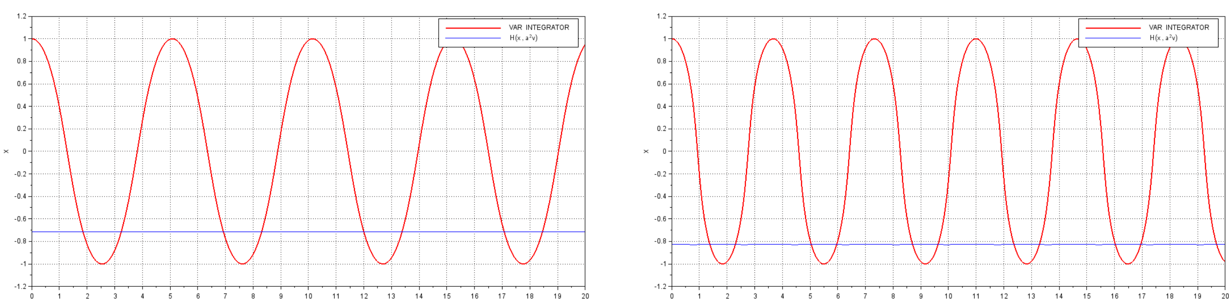


Figure IX.6: Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ with the corresponding evaluation of the first integral - Variational integrator - $h = 0.01$

IX.3.5 Topological integrator

As we want to preserve the first integral, we have to satisfy the following equation for all $i = 0, \dots, N$ as precisely as possible in the backward case :

$$\frac{1}{2} a_i^2 \left(\Delta_-[x]_i\right)^2 - \frac{1}{2} l^2 (\sin(x_i))^2 - \cos(x_i) = c_0, \tag{IX.3.23}$$

where c_0 is fixed as long as x_0 and x_1 are given. Another equivalent formulation is

$$\frac{1}{2}a_i^2(x_i - x_{i-1})^2 - h^2\frac{1}{2}l^2(\sin(x_i))^2 - h^2\cos(x_i) = h^2c_0, \tag{IX.3.24}$$

In the forward case, we have to satisfy the equation

$$\frac{1}{2}a_i^2\left(\Delta_+[x]_i\right)^2 - \frac{1}{2}l^2(\sin(x_i))^2 - \cos(x_i) = c_0, \tag{IX.3.25}$$

or equivalently that

$$\frac{1}{2}a_i^2(x_{i+1} - x_i)^2 - h^2\frac{1}{2}l^2(\sin(x_i))^2 - h^2\cos(x_i) = h^2c_0, \tag{IX.3.26}$$

This last equation can be used to replace directly the term $(x_{i+1} - x_i)^2$ in the right hand side of the forward variational integrator. Indeed, multiplying the forward variational integrator by a_i, we obtain:

$$\begin{aligned} a_i^3x_{i+1}a_i(a_i^2 + a_{i-1}^2)x_i - a_ia_{i-1}^2x_{i-1} + a_i^2l^2\sin(x_i)(x_{i+1} - x_i)^2 \\ - h^2a_i^2\sin(x_i), \quad i = 1, \ldots, N-1. \end{aligned} \tag{IX.3.27}$$

Replacing $a_i^2(x_{i+1} - x_i)^2$ by its expression, we have

$$\begin{aligned} a_i^3x_{i+1} = a_i(a_i^2 + a_{i-1}^2)x_i - a_ia_{i-1}^2x_{i-1} \\ + l^2\sin(x_i)\left(h^2l^2(\sin(x_i))^2 + 2h^2\cos(x_i) + 2h^2c_0\right) \\ - h^2a_i^2\sin(x_i), \quad i = 1, \ldots, N-1. \end{aligned} \tag{IX.3.28}$$

We then obtain the following topological integrator:

Lemma IX.1. *The topological integrator associated to the Eringen's nonlocal elastica is the explicit numerical scheme defined by*

$$\begin{aligned} x_{i+1} = \frac{1}{a_i^3}\Big[a_i(a_i^2 + a_{i-1}^2)x_i - a_ia_{i-1}^2x_{i-1} \\ + l^2\sin(x_i)\left(h^2l^2(\sin(x_i))^2 + 2h^2\cos(x_i) + 2h^2c_0\right) \\ - h^2a_i^2\sin(x_i)\Big], \quad i = 1, \ldots, N-1. \end{aligned} \tag{IX.3.29}$$

Of course, recovering an explicit numerical scheme has a price: we have destroyed the discrete variational structure of the variational integrator. Nevertheless, as we will see in the next Section, we obtain a numerical integrator with good properties in particular for the preservation of the first integral.

IX.3.6 Simulations of the topological integrator

Using the topological integrator which corresponds to the variational integrator associated to the modified equation, we obtain the following result for the same values of l:

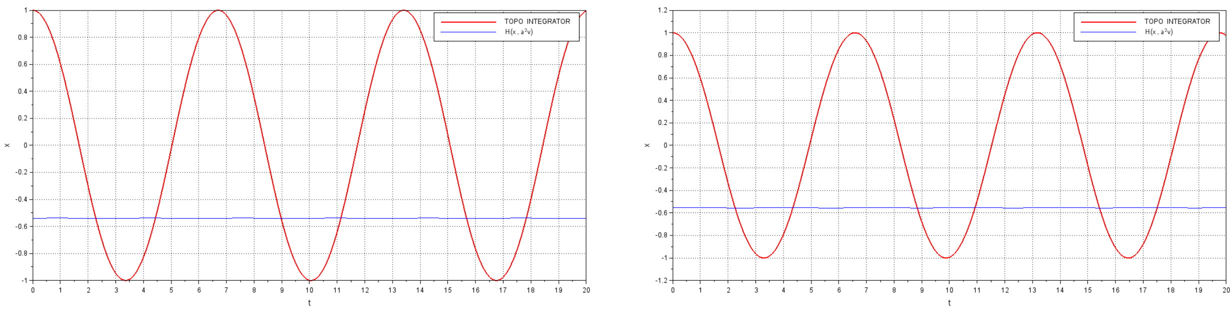


Figure IX.7: Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Topological integrator - $h = 0.01$

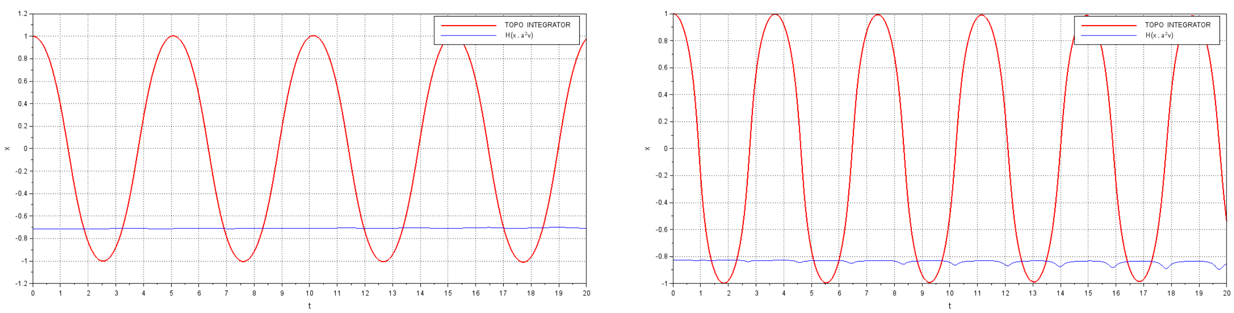


Figure IX.8: Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Topological integrator - $h = 0.01$

We have a controlled fluctuation around the exact value of the first integral $I_{1,l}$ which is a characteristic property of variational integrators due to their symplectic character.

IX.4 The Challamel's integrator

In ([23], Equation (38) p.132), N. Challamel and al. introduce a discrete version of the Eringen's nonlocal elastica by rewriting first the second order equation as a two dimensional system of first order differential equations.

Definition IX.1 (Challamel's integrator)**.** *The Challamel's integrator is defined for $i = 0, \dots, n-1$, by*

$$\begin{aligned} x_{i+1} &= x_i + h\kappa_i, \\ \kappa_{i+1} &= \kappa_i - h\beta \sin(x_{i+1}) \left(1 + l^2\kappa_i^2\right) a_{i+1}^{-1}. \end{aligned} \tag{IX.4.1}$$

In the context of the study of Eringen's nonlocal elastica, they have to consider boundary conditions given by $\kappa_0 = 0$ and $\kappa_n = 0$.

Putting aside the boundary conditions, we look for the behavior of the previous integrator with respect to the first integral obtained for the continuous Eringen's nonlocal elastica.

IX.4.1 Simulations of the Challamel's integrator

We implement the semi-implicit numerical scheme proposed by N. Challamel and al. in [23] called the Challamel's integrator in the following. We first take $h = 0.1$.

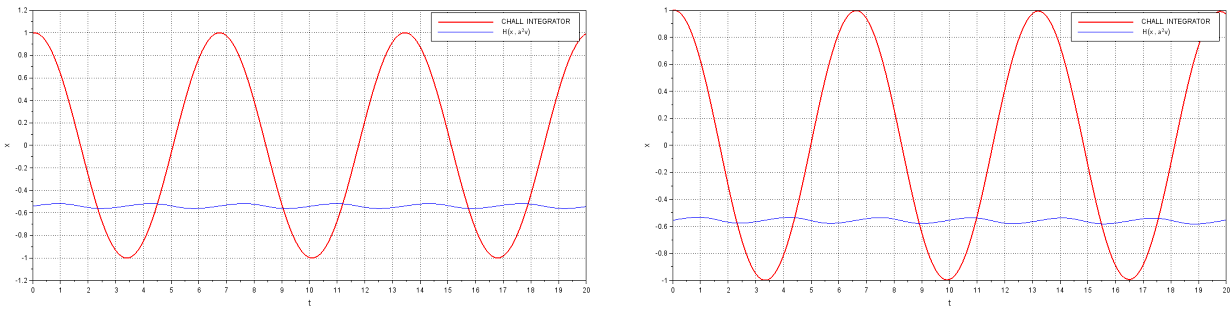


Figure IX.9: Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.1$

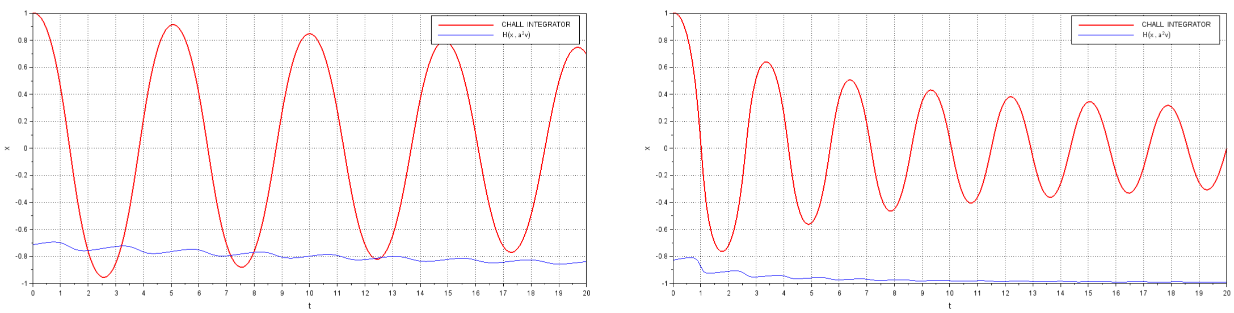


Figure IX.10: Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.1$

As one can see, we have a bad behavior for the discrete model when l is greater than 0.5. For $h = 0.01$, we obtain the following results:

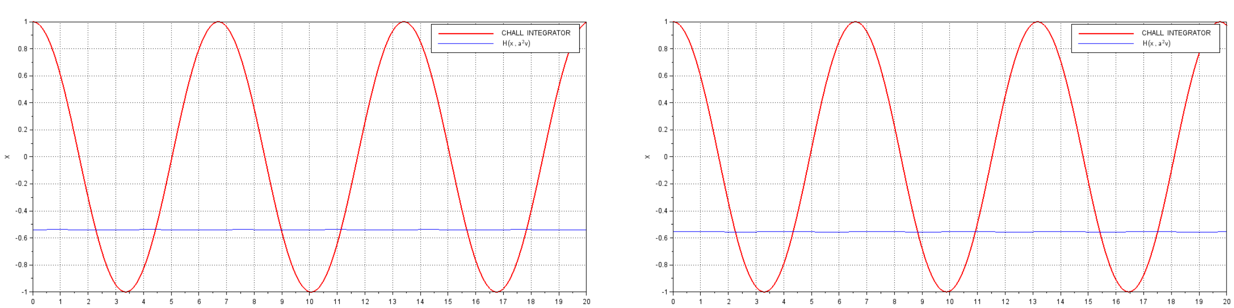


Figure IX.11: Numerical solution of the Eringen's nonlocal elastica for $l = 0$ and $l = 0.2$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.01$

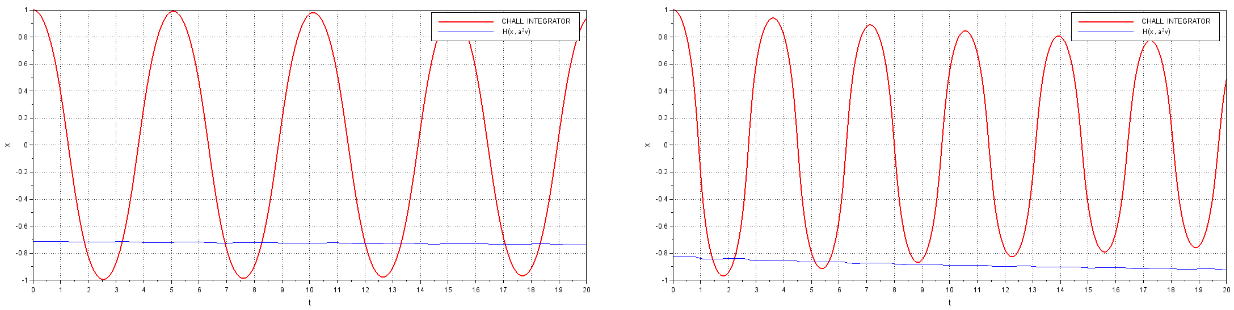


Figure IX.12: Numerical solution of the Eringen's nonlocal elastica for $l = 0.7$ and $l = 0.9$ and the corresponding evaluation of the first integral - Challamel's integrator - $h = 0.01$

Here again for large value of l the Challamel's integrator behaves badly with respect to the preservation of the first integral. The opportunity to consider this model as a good discrete analogue of the Eringen's nonlocal elastica is then questionable.

IX.5 Discrete Hamiltonian's Eringen's nonlocal elastica

The Challamel's integrator looks for a two dimensional discrete equation associated to the original second order differential equation. Having in mind that the modified Eringen's nonlocal elastica is Lagrangian, a convenient procedure is to transform the classical Euler-Lagrange equation to its Hamiltonian form as done in Section VIII.1. Using this structure, we can also derive a two dimensional discrete analogue of the modified Eringen's nonlocal elastica which preserve the Hamiltonian structure at the discrete level contrary to the Challamel's integrator. Again, we follow the discrete embedding formalism.

We recall definitions about *discrete Hamiltonian systems* in the framework of the shifted or non-shifted calculus of variations that used in Chapter VI (see Section VI.2 for more details). We restrict ourselves to uniform time scales, i.e., $\mathbb{T} = \{t_i = a + ih,\ i = 0, \ldots, N\}$ for a given $h > 0$.

Definition IX.2. *Let $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ be a function of class C^2 in each of its variables. Let $\mathbb{T}$ be a time scale. The Hamiltonian system associated to H on $\mathbb{T}^+$ is defined by*

$$\begin{cases} \Delta_+ q = \dfrac{\partial H}{\partial p}(t, q^\sigma, p), \\ \Delta_+ p = -\dfrac{\partial H}{\partial q}(t, q^\sigma, p). \end{cases} \tag{IX.5.1}$$

Using the shifted calculus of variations on time scales developed in [10], M. Bohner proved that the previous Hamiltonian systems on time scales can be obtained as critical points of shifted Lagrangian functionals on time scales. Precisely, we have:

Theorem IX.3. *The solutions of the Hamiltonian system (IX.5.1) on $\mathbb{T}$ correspond to critical points of the time scales functional*

$$\mathcal{L}^\sigma_{H,[a,b]_{\mathbb{T}}}(q, p) = \int_a^b \left[p\Delta_+ q - H(t, q^\sigma, p)\right] \Delta_+ t. \tag{$\mathcal{L}^\sigma_H$}$$

F. Pierret introduced in [85] a notion of Hamiltonian systems on time scales adapted to the framework of the non shifted calculus of variations. Precisely, we have:

Definition IX.3. *Let* $H : (t, q, p) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to H(t, q, p) \in \mathbb{R}$ *be a function of class* $\mathscr{C}^2$ *in each of its variables. Let* $\mathbb{T}$ *be a time scale. The Hamiltonian system associated to* H *on* $\mathbb{T}$ *is defined by*

$$\begin{cases} \Delta_+ q = \dfrac{\partial H}{\partial p}(t, q, p), \\ \Delta_- p = -\Delta_-[\sigma]\dfrac{\partial H}{\partial q} H(t, q, p). \end{cases} \tag{IX.5.2}$$

Here again, one can prove that Hamiltonian systems are critical point of Lagrangian functionals on time scales:

Theorem IX.4. *The solutions of the Hamiltonian systems* (IX.5.2) *on* $\mathbb{T}$ *correspond to critical points of the time scales functional*

$$\mathcal{L}_{H,[a,b]_{\mathbb{T}}}(q, p) = \int_a^b [p\Delta_+ q - H(t, q, p)]\, \Delta_+ t. \tag{$\mathcal{L}_H$}$$

We refer to the work of F. Pierret [85] for more details.

IX.5.1 Using a nonshifted discrete Hamiltonian systems for the Eringen's nonlocal elastica

Let $p = a^2(x)\Delta_+[x]$ then a discrete Hamiltonian system associated to the modified Eringen's nonlocal elastica is given by:

$$\begin{cases} \Delta_-[p] &= a\beta \sin(x)\left(l^2 \dfrac{p^2}{a^4} - 1\right), \\ \Delta_+[x] &= \dfrac{p}{a^2}. \end{cases} \tag{IX.5.3}$$

We then obtain

$$\begin{cases} p_i - p_{i-1} &= ha_i\beta \sin(x_i)\left(l^2 \dfrac{p_i^2}{a_i^4} - 1\right), \\ x_{i+1} - x_i &= \dfrac{p_i}{a_i^2}. \end{cases} \tag{IX.5.4}$$

IX.5.2 Using a shifted discrete Hamiltonian system for the Eringen's nonlocal elastica

A different version of discrete Hamiltonian systems has been introduced by C.D. Ahlbrandt, M. Bohner and J. Ridenhour in [2]. In that case, the so called shifted Hamiltonian system is given by

$$\begin{cases} \Delta_+[p] &= a(\sigma(x))\beta \sin(\sigma(x))\left(l^2 \dfrac{p^2}{a^4(\sigma(x))} - 1\right), \\ \Delta_+[x] &= \dfrac{p}{a^2(\sigma(x))}, \end{cases} \tag{IX.5.5}$$

where σ is the shift operator on the time-scale $\mathbb{T}$ defined by $\sigma(t_i) = t_{i+1}$ and as a consequence $\sigma(x_i) = x_{i+1}$.

We then obtain

$$\begin{cases} p_{i+1} - p_i &= h a_{i+1} \beta \sin(x_{i+1}) \left(l^2 \dfrac{p_i^2}{a_{i+1}^4} - 1 \right), \\ x_{i+1} - x_i &= h \dfrac{p_i}{a_{i+1}^2}. \end{cases} \tag{IX.5.6}$$

As we can see, the first equation gives an explicit formula for p_{i+1} as long as x_{i+1} is known, which relies on the resolution of the second equation which is implicit. This semi-implicit scheme is very close to the discrete Eringen's nonlocal elastica introduced by N. Challamel and al. in [23].

IX.5.3 Simulations of the shifted and nonshifted discrete Hamiltonian

In the following, we provide simulations of the shifted and non shifted discrete Hamiltonian for the Eringen's nonlocal elastica on the same figure. As we can see the difference between the two integrators is very small up to $l = 0.7$ and become ony significant for $l = 0.9$.

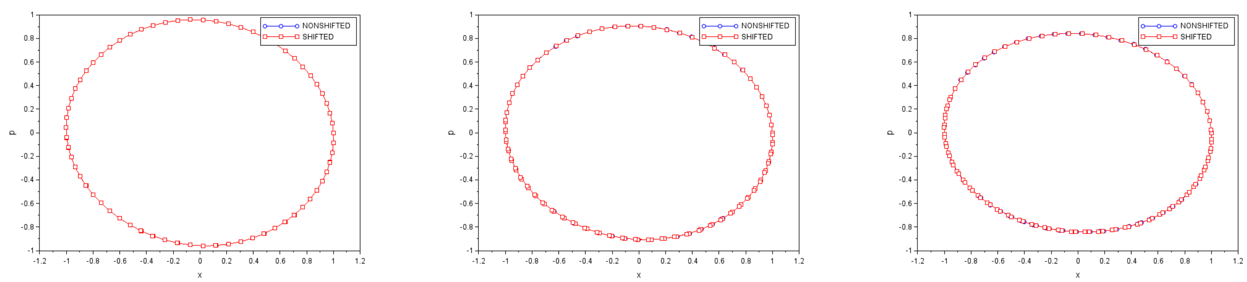


Figure IX.13: Simulations for $l = 0$, $l = 0.2$ and $l = 0.5$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

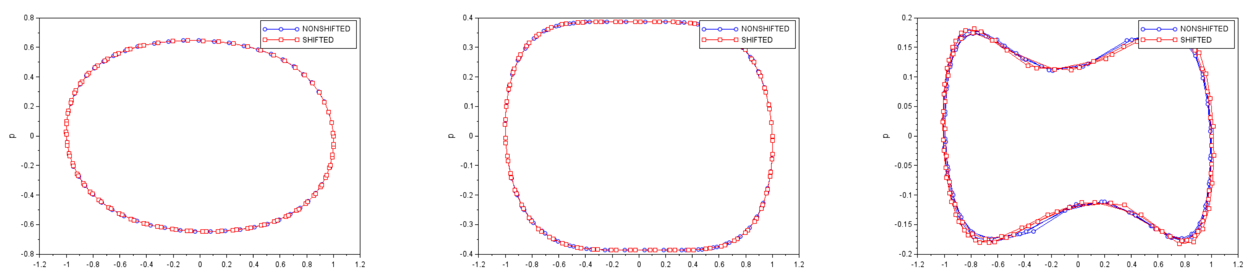


Figure IX.14: Simulations for $l = 0.5$, $l = 0.7$ and $l = 0.9$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

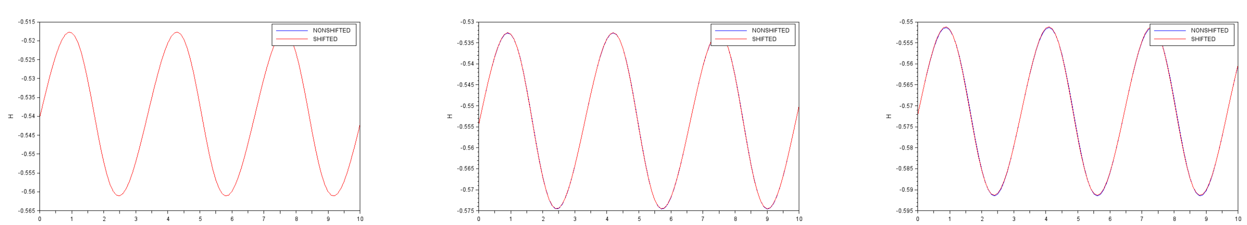


Figure IX.15: Simulations for $l = 0$, $l = 0.2$ and $l = 0.3$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

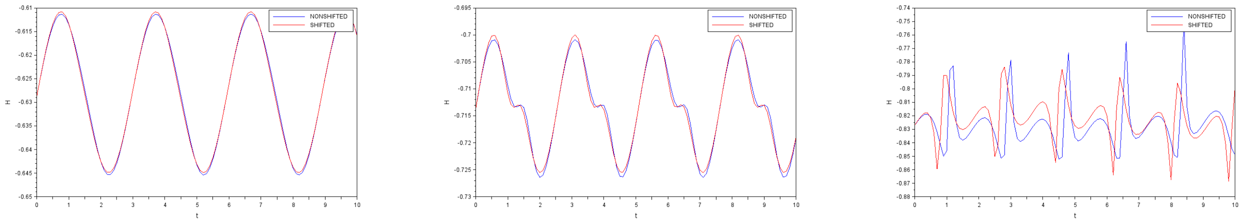


Figure IX.16: Simulations for $l = 0.5$, 0.7 and $l = 0.9$ - shifted and non-shifted Hamiltonian integrator - $h = 0.1$

IX.5.4 Comparison with the Challamel's integrator

We can compare the previous result with the one obtained using the Challamel's integrator by comparing for each integrator the behavior of the first integral $I(x) = H(x, a^2\dot{x}) = H(x,p)$. As we will see, the Challamel's integrator diverge when l increases.

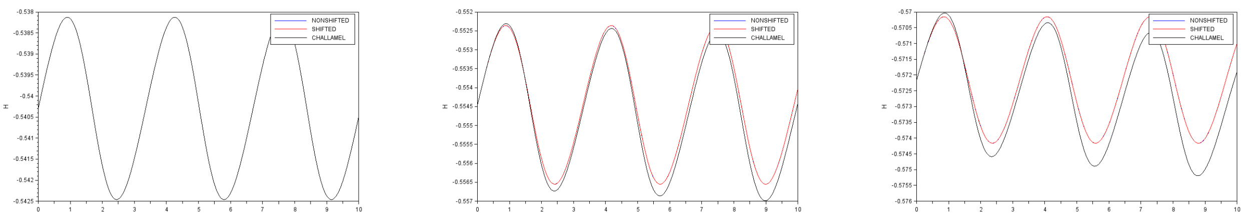


Figure IX.17: Simulations for $l = 0$, $l = 0.2$ and $l = 0.3$ -comparison of the value of the first integral for the shifted, non-shifted Hamiltonian and Challamel's integrator - $h = 0.01$

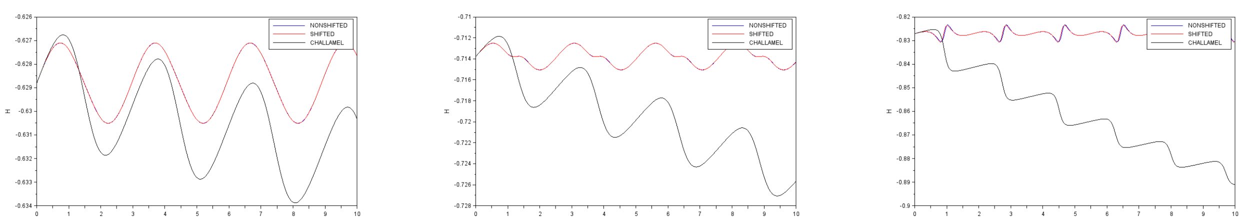


Figure IX.18: Simulations for $l = 0.5$, $l = 0.7$ and $l = 0.9$ -comparison of the value of the first integral for the shifted , non-shifted Hamiltonian and Challamel's integrator - $h = 0.01$

For each numerical scheme, we fix l and h, and we denote by $\star(t)$ the resulting discrete solution. We compute the error term as

$$e_\star(h) = \max_{t \in \mathbb{T}} \mid x(t) - \star(t) \mid, \tag{IX.5.7}$$

where x is taken as a reference solution computed for $h = 10^{-5}$.

In the following, we provide a comparison for $l = 0$, $l = 0.2$, $l = 0.7$ and $l = 0.9$ between the Euler (in green), the topological (in red), the Challamel's integrator (in blue) and the variational integrator (in magenta) for value of $h = 10^{-3}$, $h = 10^{-2}$ and $h = 10^{-1}$.

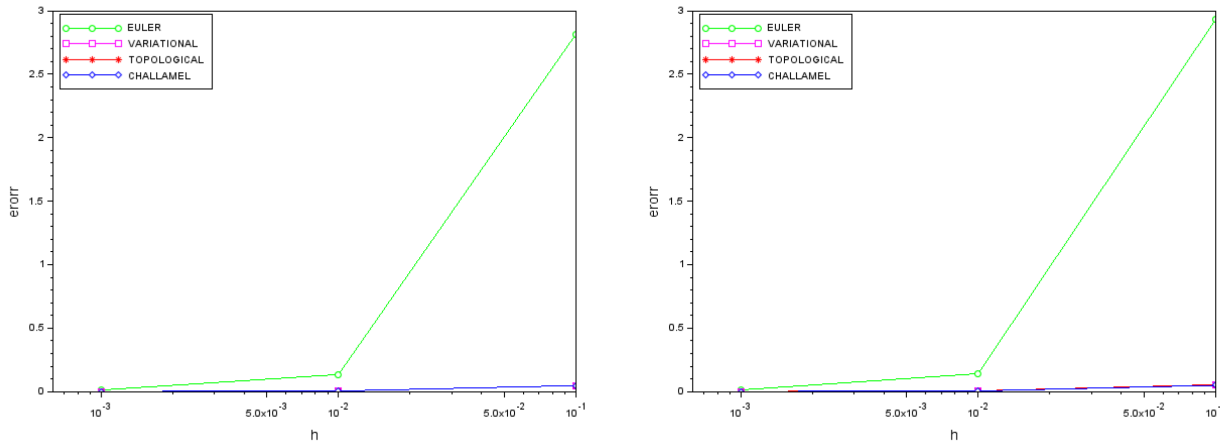


Figure IX.19: The error $e_\star$. left: $l = 0$, right: $l = 0.2$

The Euler scheme is always the less good integrator but the topological and the Challamel's one behaves more or less equally and no significant difference is observable.

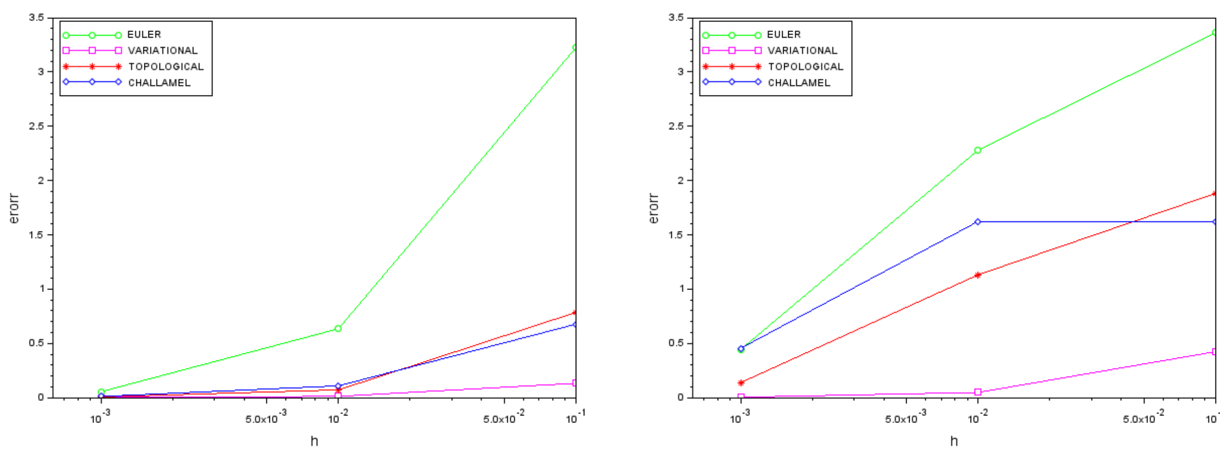


Figure IX.20: The error $e_\star$. left: $l = 0.7$, right: $l = 0.9$

As we can see the topological integrator is better than the two other when h is below 0.01 and the variational integrator gives always a better result. For greater values of l we have an increasing instability of the Euler and Challamel's integrator as can be seen in the following for $l = 0.9$.

As we can see, the variational integrator is very well adapted to the study of the Eringen's nonlocal elastica.

IX.6 Conclusion and perspectives

The previous results only give partial answers to the problems raised in [21]. As already quoted in the Introduction, this article focus on the discrete and continuous Eringen's nonlocal elastica from the point of view of their algebro-geometric structures and how they are preserved from the continuous to the discrete case. However, in order to do applications in the mechanical context, we have to take into account the boundary conditions. This will be the subject of a forthcoming paper. The explicit expression of the solutions coming from the Lagrangian and Hamiltonian structures of the modified Eringen's nonlocal elastica will be of importance for this purpose. The asymptotic behavior of the solutions as well as the bifurcation diagram will be investigated.

Chapter X

Mixing Discrete and Continuous Models

X.1 Introduction

Some physical problem can be though as discrete or continuous depending on the scale of observation. However, some physical systems (in fact, I think that this is a generic behavior) exhibit valid continuous model which do not work properly in some circumstances. In order to illustrate the problem we discuss a specific example.

X.2 Origami of Graphene

Consider an inextensible membrane on a flat substrate. Can we describe a model for **wrinkles** formation of the membrane on the substrate ?

The membrane that we consider can be think as a **Graphene** membrane for which we want to develop a practical method for constructing folds and then **origami of graphene**. A graphene is a typical discrete structure: this is an assembly of Carbon atoms which are positioned at the edges of an **hexagonal lattice**.

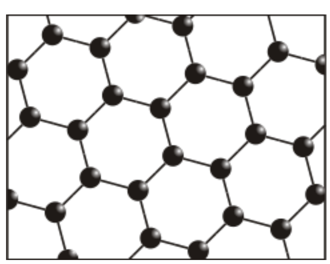


Figure X.1: Representation of a Graphene

The basic idea is first to find a procedure to create wrinkles of Graphene by compression.

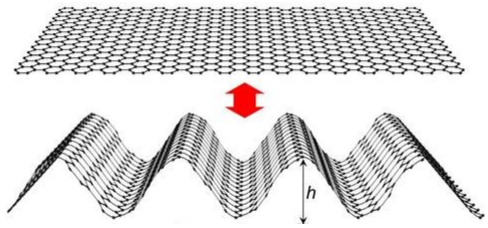


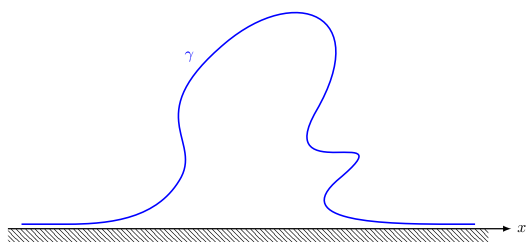
Figure X.2: Creating wrinkles of Graphene

Our main objective is to discuss the modeling of such a deformation of a Graphene plate.

X.3 A continuous model: Graphene as an inextensible membrane

A classical model used to study deformation of Graphene is a continuous one. One can identify a Graphene with a **continuous membrane** possessing some particular material properties. We refer to the article of D. Mumford ([79, p.500-501]) and the review paper of S. Matsutani [77] for an historical account of this problem as well as a derivation of the equations.

Let $(x, y, z) \in \mathbb{R}^3$ be a coordinates system. We assume for simplicity that compression will produce a deformation along the x axis in the direction z which is invariant under the perpendicular direction y, i.e., a surface of the form $(x(s), y, z(s))$ for $s \in [0, l]$ where l is the size of the wrinkle. We denote by $\gamma : [0, l] \to \mathbb{R}^2$ the mapping $s \mapsto (x(s), y(s))$.



Let $t(s)$ be the tangent vector to γ at point s, i.e.,

$$t(s) = \dot{\gamma}(s), \tag{X.3.1}$$

and $n(s)$ the normal vector to γ at point s defined by

$$n(s) = t(s)^{\perp}. \tag{X.3.2}$$

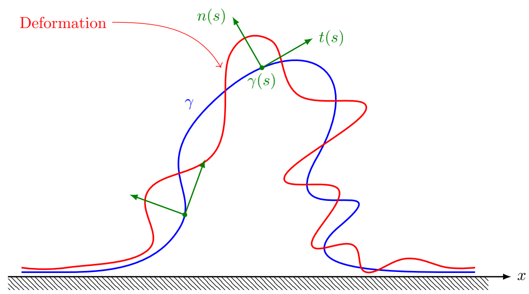
We have

$$t = \kappa n, \quad \dot{n} = -\kappa t, \tag{X.3.3}$$

where $\kappa(s)$ is the curvature of γ at point $\gamma(s)$.

We consider deformations of the curve of the form

$$\gamma_\delta(s) = \gamma(s) + \delta(s) n(s). \tag{X.3.4}$$



As a consequence, we obtain

$$\dot{\gamma_\delta} = t + \delta' n - \delta\kappa t = \langle 1 - \delta k, t + \delta' n \rangle, \tag{X.3.5}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^2$.

We look for deformation preserving the arc length, i.e.,

$$\int_{\gamma_\delta} d\tilde{s} = \int_{\gamma} ds, \tag{X.3.6}$$

where $\tilde{s}$ is the curvilinear coordinate on γ_δ.

Condition (X.3.6) is equivalent to

$$\int_\gamma \delta\kappa ds = 0. \tag{X.3.7}$$

The one dimensional profile of the membrane at equilibrium can be obtained from the energy functional defined by

$$\mathscr{L}(\gamma) = \frac{C}{2} \int_0^l \kappa^2(s)\, ds - \sigma l(h), \tag{X.3.8}$$

where s is the curvilinear coordinate on the profile Γ_γ, $\kappa(s)$ is the curvature of γ at point s, C is the curvature rigidity constant, σ is the tension and $l(\gamma)$ the length of Γ_γ between $s = 0$ and $s = l$.

By definition of the curvature and length, we have:

$$\kappa(s) = -\frac{\gamma'(s) \cdot J\gamma''(s)}{(\gamma'(s))^2)^{3/2}}, \tag{X.3.9}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{X.3.10}$$

and

$$l(\gamma) = \int_a^b \sqrt{(\gamma'(s))^2} ds. \tag{X.3.11}$$

We then introduce the Lagrangian $L : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L(x, v, w) = \frac{C}{2} \frac{(v \cdot Jw)^2}{(v^2)^3} - \sigma\sqrt{v^2}, \tag{X.3.12}$$

where we denote by $x = (x_1, x_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$ the variables and $v^2 = v_1^2 + v_2^2$.

Minimizing the energy functional (X.3.8), we obtain the following Euler-Lagrange equation

$$\frac{\partial L}{\partial x}(\star(s)) - \frac{d}{ds}\left(\frac{\partial L}{\partial v}(\star(s))\right) + \frac{d^2}{ds^2}\left(\frac{\partial L}{\partial w}(\star(s))\right) = 0, \tag{X.3.13}$$

where $\star(s) = (\gamma(s), \gamma'(s), \gamma''(s))$.

As

$$\partial_x L = 0, \quad \partial_v L = C\frac{Jw\,(v\cdot Jw)}{(v^2)^3} - 3C\frac{(v\cdot Jw)^2 v}{(v^2)^4} + \sigma\frac{v}{\sqrt{v^2}}, \quad \partial_w L = C\frac{Jv\,(v\cdot Jw)}{(v^2)^3}, \tag{X.3.14}$$

we obtain

$$-\frac{d}{ds}\left[C\frac{J\gamma''\,(\gamma'\cdot J\gamma'')}{(\gamma'^2)^3} - 3C\frac{(\gamma'\cdot J\gamma")^2\gamma'}{(\gamma'^2)^4} + \sigma\frac{\gamma'}{\sqrt{\gamma'^2}} - C\frac{d}{ds}\left[\frac{J\gamma'\,(\gamma'\cdot J\gamma'')}{(\gamma'^2)^3}\right]\right] = 0. \tag{X.3.15}$$

X.4 Breaking of the continuous model: high curvature

A basic question is: What is the validity of such a continuous model ?

In fact, it has been shown [72] that the previous continuous model is valid as long as the curvature of the underlying Graphene membrane is less or equal to a critical value denoted κ_{crit} in the following. In such a case, the microscopic structure of the membrane must be taken into account. This microscopic structure is characterized by a **characteristic scale** denoted h in the following corresponding to the mean distance between two atoms in the structure.

As a consequence, when we consider a membrane γ_0 which is plane at the beginning of the deformation, the previous model can be used. During the process appears one time t_{crit} for which the curvature of the membrane γ_t possess at least one point where the curvature $\kappa(\gamma_t)$ is greater or equal to κ_{crit}.

One point is not sufficient in order to jump to a discrete modeling. Indeed, as the membrane has a characteristic scale h, we consider that a phenomenon which appears at a scale, i.e., over a portion of the continuous membrane representation, less than h can not be covered by a discrete model. Assuming that the critical curvature is obtained over a length $l > h$, one have to replace this part by a discrete model. We denote by $t_{\text{crit},h}$ the first time were such a configuration is obtained for γ_t.

How to proceed for the construction of the discrete model ?

We propose a construction in the following section.

X.5 Building a mixed continuous/discrete model

A portion of the membrane is obtained between two curvilinear coordinates $[a,b]$ with $b-a > h$. We then introduce a time scale $\mathbb{T}_{[a,b],h}$ over $[a,b]$ such that $t_0 = a$, $t_1 = a+h$ and $t_i = a+ih$ for i such that $a+ih < b$. We denote by $N-1$ the maximal value of i and by $t_N = b$. The mapping $\gamma_{t_{\text{crit},h}} : \mathbb{T}_{[a,b],h} \to \mathbb{R}$ produce $N-1$ points between $\gamma_{t_{\text{crit},h}}(a)$ and $\gamma_{t_{\text{crit},h}}(b)$.

The idea is then to replace the continuous model over $\mathbb{T}_{[a,b],h}$ be a discrete one, by returning to a description of the mechanical behavior between atoms using elastic forces between them, then recovering a continuous model for the vibrating string or more precisely a discrete version of the continuous Eringen's nonlocal elastica obtained by N. Challamel in [22] and discussed in **hariz-cresson**. In such a configuration, we then have:

- A continuous model over $[0, l] \setminus [a, b]$,
- A discrete model over $\mathbb{T}_{[a,b],h}$.

Of course, the previous modeling can evolve with time as other parts of the continuous representative membrane can reach the critical observation scale for the curvature.

Such a mixing between a continuous and a discrete modeling can be manage by using **time-scale calculus** which was introduced by S. Hilger in 1988 (see [51], [52]). We refer to the book [2] for an overview.

It seems clear that most of material exhibit phenomena which are very similar to the previous problem of the deformation of a Graphene structure. The development of tools to deal efficiently with such dynamical systems mixing continuous and discrete systems is then need.

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Abstract

High-order embedding formalism, Noether's theorem on time scales and Eringen's nonlocal elastica

The aim of this thesis is to deal with the connection between continuous and discrete versions of a given object. This connection can be studied in two different directions: one going from a continuous setting to a discrete analogue, and in a symmetric way, from a discrete setting to a continuous one. The first procedure is typically used in numerical analysis in order to construct numerical integrators and the second one is typical of continuous modeling for the study of micro-structured materials.

In this manuscript, we focus our attention on three distinct problems. In the first part, we propose a general framework precising different ways to derive a discrete version of a differential equation called discrete embedding formalism. More precisely, we exhibit three main discrete associate: the differential, integral or variational structure in both classical and high-order approximations. The second part focuses on the preservation of symmetries for discrete versions of Lagrangian and Hamiltonian systems, i.e., the discrete analogue of Noether's theorem. Finally, the third part applies these results in mechanics, i.e., the problem studied by N. Challamel, Kocsis and Wang called Eringen's nonlocal elastica equation which can be obtained by the continualization method. Precisely, we construct a discrete version of Eringen's nonlocal elastica then we study the difference with Challamel's proposal.

Keywords: Discrete embedding, variational integrators, high-order calculus, time scale calculus, calculus of variations, group of symmetries , Lagrangian and Hamiltonian systems, Euler-Lagrange equation, Noether's theorem, constant of motions, variational principle, variational integrating factor, Eringen's nonlocal elastica.

Résume

Formalisme de prolongement d'ordre élevé, théorème de Noether sur des échelle de temps et élasticité nonlocale d'Eringen.

Le but de cette thèse est de traiter la connexion entre les versions continues et discrètes d'une équation différentielle dans deux directions. Cette connexion peut être étudiée dans deux directions différentes : l'une allant d'un cadre continu à un analogue discret, et symétriquement, d'un cadre discret à un continu. La première procédure est généralement utilisée en analyse numérique pour construire des intégrateurs numériques et la deuxième est typique de la modélisation continue pour l'étude des matériaux micro-structurés.

Dans ce manuscrit, nous concentrons notre attention sur trois problèmes distincts. Dans la première partie, nous proposons un cadre général précisant différentes manières pour dériver une version discrète d'une équation différentielle appelée formalisme de plongement discrets. Plus précisément, nous exhibons trois principaux discrets associés: la structure différentielle, intégrale ou variationnelle dans les approximations classiques et d'ordre élevé. La deuxième partie se concentre sur la préservation des symétries pour des systèmes lagrangiens et hamiltoniens discrètes, c'est-à-dire, l'analogue discret du théorème de Noether. La troisième partie, finalement, applique ces résultats en mécanique, c'est-à-dire, le problème étudié par N. Challamel, Kocsis et Wang appelé l'équation élasticité non locale d'Eringen qui peut être obtenu par la méthode de continualization. Précisément, nous construisons une version discrète de l'élasticité non locale d'Eringen puis nous étudions la différence avec le schéma proposé par Challamel.

Mots clés: Plongement discret, intégrateurs variationnelle, calcul d'ordre élevé, calcul time scale, calcul des variations, groupe de symétries, systèmes lagrangien et hamiltonien, équation d'Euler-Lagrange, théorème de Noether, constante de mouvement, principe variationnel, facteur intégrant variationnel, élasticité non locale d'Eringen,

ملخص

شكليات التضمين ذات الرتب العليا، نظرية نويثر على مقاييس الزمنية ومعادلة المرونة غير المحلية لإيرينغن

الهدف من هذه الأطروحة هو التعامل مع العلاقة بين النسخ المستمرة والمقطعنة لكائن رياضي معين. يمكن دراسة هذا الاتصال باستعمال إتجاهين مختلفين: من الوضعية المستمرة إلى المقطعنة، وبطريقة متماثلة، من الوضعية المنفصلة إلى المستمرة. الإجراء الأول شائع الإستخدام في التحليل العددي من أجل إنشاء مخططات عددية لمعادلة تفاضلية، أما الإجراء الثاني فيستخدم للنمذجة المستمرة لدراسة الهياكل الميكانيكية ذات البنية الدقيقة.

في هذه المخطوطة ، نركز اهتمامنا على ثلاث مشاكل متميزة. في الجزء الأول، نقترح إطارًا عامًا يحدد طرقًا مختلفة لاشتقاق مخطط عددي لمعادلة تفاضلية تسمى شكليات التضمين المقطعنة (discrete embedding). بتعبير أدق، نعرض ثلاث أشكال رئيسيين: الشكل التفاضلي، التكاملي أو البنية التغايرية في كل من التقريبات الكلاسيكية والعالية الدقة. يركز الجزء الثاني على الحفاظ على التناظرات بالنسبة للنسخ المقطعنة للأنظمة اللاغرنجيانية والهاملتونيانية، أي النسخة المقطعنة لنظرية نويثر. أخيرًا، الجزء الثالث هو تطبيق النتائج السابقة في الميكانيك، أي تطبيقاتها على المسألة التي درست من قبل N. Challamel و Kocsis و Wang تسمى معادلة المرونة غير المحلية لإيرنغن (Eringen) والتي يمكن الحصول عليها بطريقة (continualization). على وجه التحديد ، نقوم بإنشاء نسخة مقطعنة لمعادلة إيرنغن المحلية ثم نقارنها بالأخرى المقترحة من قبل N. Challamel

الكلمات المفتاحية: التضمين المقطعن، التكامل المتغير ، الحساب ذو الرتب العليا ، الحساب على المقاييس الزمنية ، حساب التغايرات ، زمر التناظرات ، الأنظمة اللاغرانجيانية والهاملتونيانية ، معادلة أويلر-لاغرانج ، نظرية نويثر ، ثابت الحركات ، مبدأ التباين ، العامل التكاملي ، معادلة المرونة غير المحلية لإيرنغن.