Parallel Branch-and-Price Algorithms for the Single Machine Total Weighted Tardiness Scheduling Problem with Sequence-Dependent Setup Times

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Abstract

Scheduling problems occur in a broad range of real-world application fields and have attracted a huge set of research articles. However, there is only little research on exact algorithms for scheduling problems, many of which are NP-hard in the strong sense. We investigate the problem on a single machine with a total weighted tardiness objective function and sequence-dependent setup times. First, we adopt a serial branch-and-price algorithm from the literature and present a modified branching strategy and a primal heuristic. Second, we use the potential of parallel computing architectures by presenting two parallel versions of the branch-and-price algorithm. Third, we conduct extensive computational experiments to show that our parallelization approaches provide substantial parallel speedups on well-known benchmark instances from the literature. We further observe that the parallel speedups achieved by our parallel algorithms are very robust among all tested instances.

Keywords: single machine scheduling, weighted tardiness, sequence-dependent setup times, branch-and-price algorithm, dynamic programming, shared-memory parallelization

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1. Introduction

In this study, a single machine scheduling problem with sequence-dependent setup times and a total weighted tardiness objective function is examined. This scheduling problem can be classified as $1 \mid ST_{sd} \mid \sum w_j T_j$ in the established $\alpha \mid \beta \mid \gamma$ notation by Graham et al. (1979). We refer to this problem as the *Weighted Tardiness Scheduling Problem with Sequence-Dependent Setup Times* (WTSDS). The scheduling problem $1 \mid\mid \sum w_j T_j$ without setup times is already NP-hard in the strong sense (Tasgetiren et al., 2009; Lawler, 1977). Since it is a special case of WTSDS (by setting all setup times to zero), WTSDS is strongly NP-hard as well.

Problem WTSDS can be described as follows: a set of jobs has to be processed without disruption by a single machine, which can only process one job at a time. Each job has a processing time, a due date and a weight. For every pair of jobs, processed in direct succession on the machine, a setup time is incurred. Since these setup times depend on both jobs, they are called sequence-dependent. When the completion time of a job is later than its due date, it is called tardy, with the tardiness set to the time elapsed between the due date and the time of completion. The goal of WTSDS is to find a feasible schedule with minimum total weighted tardiness.

There already exist many studies addressing the WTSDS, see Allahverdi et al. (1999, 2008), and Allahverdi (2015) for an overview. However, only a few articles present exact algorithms for WTSDS or one of its generalizations, see Section 2 for details. Since none of these papers report optimal solutions to large (e.g., more than 100 jobs) WTSDS instances, such instances are still computationally intractable.

In our paper, we adopt a serial branch-and-price (b&p) algorithm – a special kind of a branch-and-bound (b&b) algorithm where linear relaxations are solved by column generation – which was originally proposed by Lopes and de Carvalho (2007) for a generalization of WTSDS – and present a modified branching strategy which exploits problem specific sequencing information obtained from the column generation process. We also develop a primal heuristic in order to derive feasible schedules from the fractional solutions of the linear relaxations. Using this sequential b&p algorithm, we present two parallelization approaches to utilize the computing power of modern parallel hardware (Hager and Wellein, 2010). The scalability of our parallel algorithms (and limits thereof) is tested using established benchmark instances introduced by Cicirello (2003), Gagné et al. (2002) as well as Rubin and Ragatz (1995). Our computational experiments show that the proposed parallelization strategies provide substantial speedups with very little variations for the instances mentioned above.

Unlike most studies that parallelize b&b algorithms by concurrently solving nodes in the b&b tree, our study focuses on using CPU cores to parallelize the column generation process in individual b&b nodes, which has not been studied for the underlying problem. Gaining insight into the computational behavior of this type of algorithmic parallelization is valuable for a joint application of both parallelization approaches in future research, not only for the studied problem WTSDS, but also for its extensions.

The remainder of this paper is structured as follows. In Section 2, we provide a comprehensive overview of literature on machine scheduling as well as parallel branch-and-x (b&x) and dynamic programming algorithms.[1] The mathematical formulation of WTSDS is proposed in Section 3. In Section 4, a serial b&p algorithm for the WTSDS is presented. Then, we introduce two parallelization strategies for the b&p algorithm in Section 5. Detailed computational experiments of the proposed parallel algorithms are presented and discussed in Section 6. Finally, Section 7 concludes.

[1]The latter is used to solve the so called *pricing problem* during column generation in our b&p algorithms.

2. Literature Review

2.1. Machine Scheduling

Various types of machine scheduling problems have been studied by numerous authors. Comprehensive reviews covering machine scheduling problems with setup times were composed by Allahverdi et al. (1999, 2008) and Allahverdi (2015). Using the three-field notation $\alpha \mid \beta \mid \gamma$ by Graham et al. (1979), we focus on scheduling problems on single machines ($\alpha = 1$) and parallel machines ($\alpha = P$ for identical machines, $\alpha = Q$ for uniform machines and $\alpha = R$ for unrelated machines) with sequence-dependent setup times ($\beta = ST_{sd}$), and objective functions γ that are at least as general as the total weighted tardiness.

There exist a variety of (meta-)heuristic approaches for these problems, among them Ant Colony Optimization (Liao and Juan, 2007; Anghinolfi and Paolucci, 2008), Evolutionary Algorithms (Xu et al., 2014b), Iterated Local Search (Subramanian et al., 2014; Xu et al., 2014a; Subramanian and Farias, 2017), Scatter Search (Bożejko, 2010; Guo and Tang, 2015) and Variable Neighbourhood Search (Chen, 2019; Nogueira et al., 2022). Kramer and Subramanian (2019) provide an overview of solution approaches for various earliness-tardiness scheduling problems, including WTSDS. Although (meta-)heuristic approaches are often able to produce high-quality solutions, they generally don't come with any guarantee of finding optimal solutions – let alone proving their optimality. Especially since proving optimality often times takes up most of the running time for b&b algorithms (*tailing-off*), a comparison regarding running-times would be skewed. Hence, we focus on exact algorithms in the following.

Balakrishnan et al. (1999), Zhu and Heady (2000), and Akyol and Bayhan (2008) considered the total weighted earliness-tardiness objective function ($\sum w'_j E_j + \sum w''_j T_j$), which coincides with the total weighted tardiness when all earliness weights w'_j are set to zero. Balakrishnan et al. (1999) modeled the problem $Q \mid ST_{sd} \mid \sum w'_j E_j + \sum w''_j T_j$, while Zhu and Heady (2000) and Akyol and Bayhan (2008) addressed the problem $R \mid ST_{sd} \mid \sum w'_j E_j + \sum w''_j T_j$ by *Mixed-Integer Programs* (MIPs). The former two were able to solve instances with up to 12 (9 respectively) jobs and 3 machines using a commercial solver. The latter were also not able to solve larger instances with their exact algorithm based on artificial neural networks.

Lin and Hsieh (2014) present a MIP model for $R \mid ST_{sd} \mid \sum w_jT_j$ and use a commercial solver to solve instances with up to 12 jobs on 3 machines. Tavakkoli-Moghaddam and Aramon-Bajestani (2009), Lopes and de Carvalho (2007), and Lopes et al. (2014) solved the same scheduling problem by presenting b&b and b&p algorithms based on MIP formulations. The former were capable of solving instances with up to 10 jobs and 4 machines, while the latter two were able to solve larger instances with up to 150 (180 respectively) jobs and 50 machines within slightly more than one hour (four hours respectively).

Regarding WTSDS, Tanaka and Araki (2013) implemented an exact algorithm capable of solving instances with up to 85 jobs within a maximum of 34 days, while Nogueira et al. (2019) proposed several MIP formulations which were used to solve instances with up to 20 jobs within one hour. Sewell et al. (2012) considered the problem without weights, $1 \mid ST_{sd} \mid \sum T_j$, and developed an exact algorithm able to solve instances with up to 45 jobs within half an hour.

In addition to the paucity of research on exact algorithms for machine scheduling problems, none of the exact approaches cited in this Section exploits the opportunities of parallel computation. In our paper, we address both of these issues by adopting the algorithm of Lopes and de Carvalho (2007) for WTSDS, modifying its branching strategy, developing a primal heuristic to obtain upper bounds more frequently, and presenting and computationally evaluating parallelized versions of the adapted algorithm.

2.2. Parallel Branch-and-X Algorithms

Parallel b&x algorithms have been applied to various types of problems. The list of considered problem types comprises assignment problems (Galea and Le Cun (2011)), graph theory problems (e.g., Christou and Vassilaras (2013)), knapsack problems (e.g., Ismail et al. (2014)), mixed integer linear programs (e.g., Carvajal et al. (2014)), stochastic optimization problems (e.g., Aldasoro et al. (2017)), and traveling salesman problems (e.g., Ozden et al. (2017)). Furthermore, flow shop scheduling problems (e.g., Chakroun et al. (2013a,b)), job shop scheduling problems (e.g., Adel et al. (2016); AitZai and Boudhar (2013)), parallel machine scheduling problems (Rauchecker and Schryen (2015, 2018)) and the problem $1 \,||\, \sum T_i$ (Wodecki, 2008) are considered; for a comprehensive overview of the application of parallel b&x algorithms to optimization problems, see Schryen (2020).

According to the framework for parallel optimization in operations research introduced by Schryen (2020), there are three types of parallelization strategies for b&x algorithms. These are based on the taxonomy of Crainic and Toulouse (2003) and Gendron and Crainic (1994). In the first strategy, which is based on domain decomposition, the solution space is split, and the partitions are explored concurrently, which for b&x algorithms means that multiple active nodes of the b&b tree are explored simultaneously. For this concept, Schryen (2020) introduced the term *coarse-grained intra-algorithm* parallelism. In coarse-grained inter-algorithm parallelism, the second strategy, the solution space is not decomposed, but multiple b&x procedures are executed concurrently on the same solution space. In the third strategy, which Schryen (2020) calls *fine-grained intra-algorithm* parallelism, only a small, predefined part of the algorithm is parallelized when exploring a local region. Hence, this is also called a low-level strategy. This entails that the computation of an operation at a single b&b node is executed in parallel.

Rauchecker and Schryen (2015) implemented a fine-grained intra-algorithm parallelization strategy in which solving the pricing problem and the branching decision is parallelized. Chakroun et al. (2013b) chose the same parallelization strategy by computing the lower bound of each node of the tree in parallel on graphics processing units (GPUs). Carvajal et al. (2014) chose a coarse-grained inter-algorithm parallelization strategy. They execute several configurations of a solver in parallel, each applying a b&b procedure on the same mixed-integer linear programming problem while sharing information with each other.

In all other studies on parallel b&x algorithms (based on the review by Schryen (2020)), a coarse-grained intra-algorithm parallelization strategy is realized (e.g., Aldasoro et al. (2017), Christou and Vassilaras (2013), Ismail et al. (2014), Ozden et al. (2017), and Rauchecker and Schryen (2018)). This is the most natural, since most straightforward, way to parallelize b&x algorithms (Schryen, 2020). Galea and Le Cun (2011) combined the coarse-grained with a fine-grained intra-algorithm parallelization strategy in a hybrid approach. In their article, the computation of the lower bound is vectorized and then parallelized by means of single instruction, multiple data (SIMD) instructions. Adel et al. (2016) also implemented a two-level parallelization on a GPU. On the one hand, the computation of the lower bound at each node is executed in parallel, on the other hand, multiple nodes are explored in parallel. These examples also show that fine-grained and coarse-grained intra-algorithm parallelism are not mutually exclusive and can be combined in hybrid approaches.

Regarding scheduling problems, there appears a high diversity of parallelization strategies. Chakroun et al. (2013b) and Rauchecker and Schryen (2015) use a fine-grained intra-algorithm parallelization strategy for a flow shop scheduling problem and a parallel machine scheduling problem, respectively. Chakroun et al. (2013a), Chakroun and Melab (2015), Gmys et al. (2016, 2017), Mezmaz et al. (2014), and Vu and Derbel (2016) apply coarse-grained intra-algorithm parallelization to flow shop scheduling problems. AitZai and Boudhar (2013) and Rauchecker and Schryen (2018) also present coarse-grained intra-algorithm parallelization strategies for a job shop scheduling problem and a parallel machine scheduling problem, respectively. Finally, Adel et al. (2016) introduce a hybrid approach of coarse-grained and fine-grained intra-algorithm parallelization for a job shop scheduling problem.

In our paper, we apply a fine-grained intra-algorithm parallelization to the (single) machine scheduling problem WTSDS by parallelizing the processing of linear relaxations at the b&p tree nodes. In particular, we parallelize the *dynamic programming* algorithm, which is the most time-consuming part of the column generation procedure used to solve the linear relaxations. As this entails solving a kind of shortest path problem, this approach is not limited to the problem WTSDS, but could be adapted to a variety of other algorithms for optimization problems that involve solving a similar dynamic programming problem. Acknowledging that parallelizing b&x algorithms through a coarse-grained intra-algorithm parallelization by solving nodes of the b&b tree concurrently is the most studied approach, we focus in this work on fine-grained intra-algorithm parallelization. Once the individual effects of the fine-grained parallelization are analyzed, without interfering effects such as early pruning of b&b nodes due to processing them in parallel, a hybridization can be approached in future research.

2.3. Parallel Dynamic Programming Algorithms

Parallel dynamic programming (DP) algorithms have also been applied to numerous types of problem. Kumar et al. (2011), Stivala et al. (2010), and Tran (2010) use parallel DP to solve graph theory problems while Boyer et al. (2012), Rashid et al. (2010), and again Stivala et al. (2010) apply parallel DP to knapsack problems. Aldasoro et al. (2015) and Rauchecker and Schryen (2015) present parallel DP algorithms for a stochastic optimization problem and a parallel machine scheduling problem, respectively. Applications of parallel DP to several other problem types are presented by Boschetti et al. (2016), Dias et al. (2013), Maleki et al. (2016), and Tan et al. (2009).

The only approach for using parallel DP as part of a column generation procedure is presented by Rauchecker and Schryen (2015) for a parallel machine scheduling problem. In their parallelization, the solution space is divided into independent subproblems, each corresponding to exactly one of the parallel machines, which are solved in parallel. This strategy is not applicable to WTSDS since there is only a single machine in WTSDS. Consequently, we present a new parallel DP approach for executing the column generation procedure in our b&p algorithm in parallel.

3. Decision Support Model

To present a decision support model for WTSDS, we rely on Lopes and de Carvalho (2007), who investigate a generalization of WTSDS. Let $\{1,\dots,n\}$ be a set of jobs which must be processed by the single machine. For each job $j \in \{1,\dots,n\}$, the due date is denoted by d_j, the processing time by p_j, and its weight by w_j. We denote the setup times between two jobs i and j by s_{ij}. A feasible schedule $\omega = (j_1,\dots,j_h)$, with $0 \le h \le n$, is a tuple of pairwise different jobs $j_1,\dots,j_h$ and represents the order in which the jobs are processed on the machine. We denote the set of all feasible schedules by Ω. The parameter $a_{j\omega} \in \{0,1\}$ indicates whether job j is contained in schedule ω. Furthermore, the weighted tardiness of a schedule ω, which is formalized at the end of this section in equation (5), is denoted by T_ω. For each schedule $\omega \in \Omega$ we introduce a binary decision variable x_ω which equals 1 if ω is operated and 0 otherwise. Consequently, WTSDS can be formulated as a binary linear program:

$$\min \quad \sum_{\omega\in\Omega} T_\omega \cdot x_\omega \tag{1}$$

$$\text{s.t.} \quad \sum_{\omega\in\Omega} a_{j\omega} \cdot x_\omega = 1 \quad \forall j = 1,\dots,n \tag{2}$$

$$\sum_{\omega\in\Omega} x_\omega \le 1 \tag{3}$$

$$x_\omega \in \{0,1\} \qquad \omega \in \Omega \tag{4}$$

The objective function (1) represents the total weighted tardiness. Constraints (2) ensure that each job is processed exactly once, while constraint (3) guarantees that at most one schedule $\omega \in \Omega$ is allowed on the machine.

The weighted tardiness T_ω of a schedule $\omega = (j_1,\dots,j_h) \in \Omega$ is defined as

$$T_\omega = \sum_{\ell=1}^{h} \max\{C_{j_\ell} - d_{j_\ell}, 0\} \cdot w_{j_\ell} \quad , \tag{5}$$

where C_{j_ℓ} denotes the completion time of job j_ℓ. Then $C_{j_1} = s_{0j_1} + p_{j_1}$, where s_{0j} represents the initial setup time to process job j first, and $C_{j_r} = C_{j_{r-1}} + s_{j_{r-1}j_r} + p_{j_r}$ for all $2 \le r \le h$.

4. A Serial Branch-And-Price Algorithm

The model presented in the previous section can be solved by a b&p algorithm. A generic description of this type of algorithm, first conceptualized by Barnhart et al. (1998), is presented in Algorithm 1.

Algorithm 1 B&p algorithm (Barnhart et al., 1998)

```
1: Solve linear relaxation of root node using column generation
2: Initialize set of active nodes
3: repeat
4:     Select an active node
5:     Branch on selected node
6:     Solve new nodes' relaxations using column generation
7:     Update set of active nodes
8: until set of active nodes is empty
```

In the first step of the algorithm (line 1), the linear relaxation of the root node is solved by column generation, which is described in Section 4.2. Regarding WTSDS instances, the root node of the b&b tree corresponds to the problem modeled in (1)–(4), where constraints (4) are relaxed to $0 \leq x_\omega \leq 1$ for all $\omega \in \Omega$ (actually, $x_\omega \geq 0$ suffices due to constraint (3)). If the optimal solution of the root node relaxation is integer, an optimal solution for WTSDS has been found. Otherwise, the set of active nodes is initialized with the root node (line 2).

After initializing the set of active nodes, lines 4–7 are repeated until there are no more active nodes left. First, one of the active nodes is selected to be branched on. Branching creates two child nodes that are added to the set of active nodes while the parent node is removed. The node selection (line 4) and the corresponding branching strategy (line 5) are specified in Section 4.1. The next step is to solve the child nodes, again using column generation (line 6). Based on their optimal solutions, the set of active nodes of the b&b tree is updated (line 7). In the course of the update, an active node is declared as inactive in three cases. First, if the selected node's relaxation is not feasible, or second, if it has an integer optimal solution. The first case cannot happen for our algorithm, since all columns are inherited from the parent node, but those which become infeasible due to branching are heavily penalized in the objective function. Regarding the third case, note that in a minimization problem, the optimal solution of a node's relaxation constitutes a lower bound on its optimal integer solution. Therefore, third, a node is declared as inactive if the solution value of the node's relaxation is greater than the value of the best known integer solution found so far in the tree, which is called *pruning* (or *fathoming*).

Once there are no active nodes left, the current best integer solution is also the optimal solution for the WTSDS instance.

4.1. Node Selection and Branching Strategy

For node selection, we apply *best-first search*, where the active node with the lowest lower bound is selected to be explored next. For branching on a selected node, we follow an established approach, see for example Chen and Powell (1999) or Akker et al. (1999), and introduce branching decision variables X_{ij}, which are defined as

$$X_{ij} = \sum_{\omega \in \Omega} \delta_{ij\omega} \cdot x_\omega \leq 1 \tag{6}$$

for every $i = 0, \ldots, n$ and $j = 1, \ldots, n$, where $\delta_{ij\omega} \in \{0, 1\}$ indicates whether job i is processed immediately before job j in schedule ω while $(x_\omega)_{\omega \in \Omega}$ is an optimal solution of the selected node's relaxation. Hereby, $\delta_{0j\omega}$ is defined as 1 if j has no predecessor i in schedule ω because j is the first job processed in ω, and 0 otherwise. In case x_ω is binary for all schedules $\omega \in \Omega$, X_{ij} indicates whether job i is processed immediately before job j or not. Accordingly, X_{0j} indicates whether job j is processed first or not. Note that $a_{j\omega} \in \{0.1\}$ in (2) and $\delta_{ij\omega} \in \{0, 1\}$ in (6) will be relaxed to $a_{j\omega}, \delta_{ij\omega} \in \mathbb{N}$ to significantly reduce the state space of the dynamic programming procedure during the pricing phase (Lopes and de Carvalho, 2007).

The branching strategy we use is a modified version of *most-fractional branching* (abbreviated by MF). While in MF a variable with a value closest to 0.5 is selected, we also use problem-specific information. Our strategy for selecting a branching edge therefore consists of two steps. First, we determine a set A of the $\frac{n\cdot(n+1)}{100}$ edges (i,j) with the lowest values for $|X_{ij}-0.5|$. In other words, these are the edges with the 1% most fractional values of X_{ij}. Second, from this set, we select the edge (i^*,j^*) with the lowest average weighted position over all feasible schedules ω, calculated by

$$(i^*,j^*) = \underset{(i,j)\in A}{\arg\min} \sum_{\omega\in\tilde{\Omega}} x_\omega \cdot \text{position}_{ij\omega} \quad , \tag{7}$$

as we want to branch on an edge which is scheduled preferably early in the most promising schedules. Note that for $\text{position}_{ij\omega}$ we only consider the first occurrence of an edge (i,j) in a schedule ω; a discussion of why there may be more than one occurrence is provided in Section 4.2. The variable $X_{i^*j^*}$ is the one that we branch on.

By branching on $X_{i^*j^*}$, job ordering restrictions are set for the two child nodes of the node that we branch on. In the first child node, $X_{i^*j^*}$ is set to 1. This implies that only schedules in which job i^* is processed immediately before j^* are permitted in this node. In the second child node, $X_{i^*j^*}$ is set to 0, which means that only schedules with job i^* not being processed directly before j^* are feasible for this node. For both child nodes, the set of allowed schedules is modified according to the imposed restrictions. Therefore, each node of the b&b tree represents a problem following the structure of model (1)-(4), only each with its node-specific set of schedules . In the following, we denote these sets by $\tilde{\Omega} \subseteq \Omega$, where each $\omega \in \tilde{\Omega}$ complies with every branching decision that led to the selected node.

4.2. Column Generation Process

In this Section, we detail the column generation procedure used to solve the linear relaxations of the b&b nodes (lines 1 and 6 in Algorithm 1). All algorithms in this Section are adopted unchanged from Lopes and de Carvalho (2007). As noted above, b&b node relaxations all have the same structure – only differing by node-specific sets of feasible schedules $\tilde{\Omega}$ – and can therefore be solved by a single column generation procedure, which we present in Algorithm 2.

Algorithm 2 Column Generation Procedure

```
1: repeat
2:     Solve a restricted form of the original LP called the restricted LP considering only a (typically small) subset of variables
3:     Solve the pricing problem for the restricted LP:
    3.1: Let $(\pi, \sigma)$ denote the optimal dual solution of the restricted LP, i.e., $\pi_j$ is the dual variable corresponding to job $j$ in constraints (2) and $\sigma$ is the dual variable corresponding to constraint (3).
    3.2: Determine if there is any variable, i.e., column, $x_\omega$ in the original LP that has a negative reduced cost
             $r_\omega = T_\omega - \sum_{j=1}^{n} a_{j\omega}\pi_j - \sigma$     (8)
4:     Add the variable $x_\omega$ with the least reduced cost to the restricted LP if the least reduced cost
             $r^* = \min_{\omega \in \tilde{\Omega}} r_\omega = \min_{\omega \in \tilde{\Omega}} \left\{ T_\omega - \sum_{j=1}^{n} a_{j\omega}\pi_j - \sigma \right\}$     (9)
       is negative, i.e., $r^* < 0$
5: until there are no more variables with negative reduced cost, i.e., an optimal solution is found
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The linear relaxation of a b&b node is referred to as the *original LP*. By considering only a small subset of variables, the original LP is initially scaled down to a so-called *restricted LP* (line 2). To build the initial restricted LP of a node in the first execution of line 2, an initial set of variables is required. For the root node, this initial set can be generated by any WTSDS heuristic. We use the solution heuristic suggested by Lee et al. (1997) for our algorithm. For all other nodes, the initial set is inherited from the parent node. Note that branching on an edge $(i^*.j^*)$ creates two child nodes, in one of which the job i^* is prohibited from directly preceding job j^*. The way the edge was calculated in (7), this leads to at least one infeasible schedule ω, where $x_\omega > 0$ holds in the solution of the parent node. To ensure a feasible linear program, schedules which become infeasible must still be accessible in the next iteration when the child node is processed. Otherwise, the dual variables needed in the pricing problem (see line 3 in Algorithm 2) cannot be computed. Hence, these schedules must not be removed from the problem immediately, but are penalized in the objective function by a sufficiently large value instead. Note that the alternative of introducing an artificial schedule $\bar{\omega}$ containing all jobs also adds the corresponding coefficients $a_{j\bar{\omega}}$ to the model, which affects the computation of the dual variables π_j and could ultimately have an impact on the column generation process. Since infeasible columns are only added to model in one more iteration, the size of the LP is not an issue.

After that, in line 3, the so-called *pricing problem* is solved, which corresponds to identifying the variable (i.e., column) x_ω with the least reduced cost. The pricing problem is solved by dynamic programming, which is described later in Algorithm 3. In line 4, the variable x_ω with the least reduced cost (if negative) is added to the restricted LP. If there are no more columns with negative reduced costs, the optimal solution obtained for the restricted LP also serves as an optimal solution for the original LP and thus for the linear relaxation of the processed b&b node.

To solve the pricing problem in line 3 of Algorithm 2, a dynamic programming algorithm, which is described in Algorithm 3, is used. For this algorithm to be viable, we have to allow for cyclic schedules, where jobs may be processed more than once in the same schedule $\omega \in \Omega$. This reduces the state space of the dynamic programming algorithm for the pricing problem from exponential to pseudo-polynomial size, since only the immediate predecessor of each job needs to be remembered. This entails that $a_{j\omega}$ and $\delta_{ij\omega}$ for jobs i and j in a schedule ω are no longer necessarily binary for linear relaxations of the WTSDS. Note that this does not affect the optimal solution of an WTSDS instance, since the

Algorithm 3 Solving the Pricing Problem

```
1: Initialize $f(t, j) = \infty$ for each time $t \leq 0$, and job $j$.
2: Initialize $f(0, 0) := -\sigma$ and $f(t, 0) = \infty$ for each time $0 \neq t \leq T$.
3: For each time $1 \leq t \leq T$, and job $j$, set
       $$f(t, j) = \min_{i \in \mathcal{P}_j} f(t - s_{ij} - p_j, i) + \max\{t - d_j, 0\} w_j - \pi_j. \quad (10)$$
4: The *minimum reduced cost under the value* $T$ is defined as
       $$r^*_T = \min_{t=0,\ldots,T} \min_{j=0,\ldots,n} f(t, j). \quad (11)$$
```

exactly-once processing of each job is ensured by constraints (2).

The set of all possible predecessors of job j, denoted by $\mathcal{P}_j$, differs for each node as it depends on node-specific ordering restrictions imposed by the branching strategy. We define $f(t, j)$ for a time $t \in \mathbb{Z}$ and a job j as the minimum reduced cost of a variable x_ω, where job j is completed at time t as the last job being processed in schedule ω. For each variable x_ω, reduced costs $f(t, j)$ for all jobs j at all times $t \leq T$ (with an upper bound T on the makespan of an integer optimal solution of the current node) are calculated recursively to identify the minimum reduced cost. Based on that, the corresponding schedule with minimum reduced cost under the time limit T can be constructed reversely.

Furthermore, we implement two methods for enhancing the efficiency. On the one hand, we start with a low value of T and successively adjust the time limit until there are no more variables with negative reduced costs for two consecutive values of T. On the other hand, we significantly reduce the solution space of the pricing problem by considering only so-called *decreasing reduced cost schedules* under certain conditions and by preventing generated schedules from containing sequences of the form (i, j, i) (so called 2-*cycles*). For details, see Lopes and de Carvalho (2007).

4.3. Primal Heuristic

After the column generation process terminates, the resulting solution of the *restricted problem* $(x_\omega)_{\omega \in \Omega}$ is fractional for most nodes of the b&b tree. Additionally, since we allow cyclic schedules to be generated, there will be infeasible schedules ω in the solution of the LP; i.e., $x_\omega > 0$ holds for such schedules. To illustrate that this phenomenon, consider an instance with jobs $1, 2, 3, 4$ and suppose the schedules $\omega_1 = (1, 2, 1, 2)$ and $\omega_2 = (3, 4, 3, 4)$ have been generated. We can confirm that $x_{\omega_1} = x_{\omega_2} = \frac{1}{2}$ is feasible for the relaxation of the set partitioning model (1)–(4). Although both of these schedules are infeasible for the original problem, they still contain sequencing information because their columns were generated based on their corresponding reduced costs. Thus, ω_1 and ω_2 indicate that it's favorable to have jobs 2 and 4 directly succeed jobs 1 and 3.

This sequence information has already been used in Section 4.1 to select the branching edge and stored in the variables X_{ij} from Equation (6). We construct a feasible schedule by iteratively using the largest values of X_{ij}; i.e., we start with the (artificial) job, set $i = 0$, schedule

$$j = \underset{k \text{ unscheduled}}{\arg\max} \; X_{ik} \tag{12}$$

next and continue this way with j as the next i until all jobs are sequenced or the maximum value of X_{ij} is zero. In the latter case, there are jobs left which have not been scheduled yet (as would be the case in the example above), and the remaining jobs are appended to the sequence in ascending order of their *(weighted) due dates* (EDD rule).

Since the X_{ij} values have already been computed prior to the primal heuristic, the above procedure is very efficient in terms of additional computation time. Thus, we follow the idea of Atakan et al. (2017) to extend the primal heuristic by a simple local search procedure. We start by iteratively considering all possible swaps of two jobs in our sequence until the best swap among them does not lead to any improvement of the current schedule's objective value (2-opt). We then apply the same procedure with three jobs at a time (3-opt). This extension is applied whenever the primal heuristic described above generates a solution which is at most 10% worse than the upper bound found so far by the b&b algorithm, or whenever an improving integer solution is found. With these restrictions, preliminary experiments showed that only a few improvements are made each time, and thus there is no need to set a tight iteration limit for the two extensions in order to reduce computation time.

When abbreviations are used to refer to an algorithm, if the primal heuristic is applied, it is indicated by the suffix "PH".

5. Parallelization of the Serial Branch-And-Price Algorithm

In this section, we present our parallelization strategies for the serial b&p algorithm presented in Section 4. We use fine-grained intra-algorithm parallelization, see Section 2, by solving the linear relaxations of individual b&p tree nodes in parallel (lines 1 and 6 in Algorithm 1). This corresponds to parallelizing the column generation procedure (Algorithm 2). We identified the dynamic programming (Algorithm 3) as the most time-consuming part of the column generation procedure, and therefore this part is chosen for parallelization.[2]

The crucial part of dynamic programming is to calculate $f(t, j)$ for all time slots $1 \leq t \leq T$ and for all jobs $0 \leq j \leq n$. Fixing $1 \leq t_0 \leq T$ and using parallel threads to calculate $f(t_0, j)$ simultaneously for all jobs j proved to be inefficient in our pretests. Therefore, we use each parallel thread for calculating all values $f(t, 0), f(t, 1), \ldots, f(t, n)$ for several $1 \leq t \leq T$. However, there are data dependencies between the values $f(t, j)$ and $f(t', j')$ for different time slots t, t' and jobs j, j', as we can see from the definition in equation (10). Hence, parallel dynamic programming may be obstructed by waiting times. Therefore, reducing

[2]The other non-trivial part of the column generation procedure is the repeated solving of restricted LPs. This can be easily parallelized by off-the-shelf solvers. However, we did not observe any substantial benefit from using solver parallelization (probably because the restricted LPs are too small for effective parallelization), and therefore we do not follow this approach in our paper.

waiting times to a minimum constitutes the prime challenge in parallelizing our dynamic programming algorithm.

Also, these dependencies lead to an instance-specific number of threads where we can expect to observe little positive (or even negative) effects from using additional threads. For an instance of WTSDS, let $\bar{T} = \max_{i,j} s_{ij} + p_j$ be the maximum amount of time that can elapse before another job can be processed, and let us assume that we use at least $\bar{T} + 1$ threads. In this scenario, the thread processing the latest time slot is exclusively accessing time slots which are all still being processed by other threads and therefore contain missing entries. This phenomenon is evaluated and discussed in Section 6.

We introduce two versions of parallel dynamic programming, which we explain in below. Both versions are implemented using the OpenMP shared-memory programming paradigm (OpenMP, 2015), which allows code to be executed in parallel on multiple shared-memory threads using the statement `#pragma omp parallel for`. We use this pragma to parallelize the outer loop $t = 0, \ldots, T$ of dynamic programming on multiple OpenMP threads.

The first version is shown in Algorithm 4 and performs what we refer to as *strict parallel dynamic programming (s-pdp)*. Whenever the reduced cost $f(t - s_{ij} - p_j, i)$ for any possible predecessor i of job j is not available at the time when the calculations for job j at the time t are trying to access the value, *s-pdp* waits for it to become available before continuing, as shown in lines 5–7 of Algorithm 4.[3]

[3]Due to potential synchronization delays, simply waiting for the values to become available is not economical. In OpenMP, a thread uses a cache where it temporarily holds data from the shared memory. For efficiency reasons, consistency between this temporary cache and the shared memory is not always given (Hoffmann and Lienhart, 2008, p.109). Therefore, in many cases, the required values may be already calculated, but may just not yet be accessible due to a lack of synchronization. Consequently, if a thread determines that a shared variable value required for its calculations is not yet available, it first updates its cache to ensure that its cached values are up-to-date. An update of the required data can be initiated by a so-called *flush* directive, which synchronizes a thread's cache and the shared memory.

Algorithm 4 Strict Parallel Dynamic Programming (s-pdp)

```
1: #pragma omp parallel for
2: for $t = 0, \dots, T$ do
3:     for $j = 1, \dots, N$ do
4:         for $i \in \mathcal{P}_j$ do
5:             while $f(t - s_{ij} - p_j, i)$ is not available do
6:                 flush $f(t - s_{ij} - p_j, i)$
7:             read $f(t - s_{ij} - p_j, i)$
8:         write $f(t, j) = \min_{i \in \mathcal{P}_j} f(t - s_{ij} - p_j, i) + \max\{t - d_j, 0\}w_j - \pi_j$
```

In our second version, dynamic programming is executed on parallel threads without verifying the values of shared variables. We refer to this as *loose parallel dynamic programming (l-pdp)*, which is described in Algorithm 5. In the case that the reduced cost $f(t - s_{ij} - p_j, i)$ is not yet available, calculations are still continued using the default value for $f(t - s_{ij} - p_j, i)$ set in line 1. Consequently, the values for $f(t, j)$ returned by *l-pdp* may be incorrect.

Algorithm 5 Loose Parallel Dynamic Programming (l-pdp)

```
1: $f(t, j) = \infty \quad \forall\, 0 \le t \le T$ and $\forall\, 1 \le j \le N$
2: #pragma omp parallel for
3: for $t = 0, \dots, T$ do
4:     for $j = 1, \dots, N$ do
5:         for $i \in \mathcal{P}_j$ do
6:             read $f(t - s_{ij} - p_j, i)$
7:         write $f(t, j) = \min_{i \in \mathcal{P}_j} f(t - s_{ij} - p_j, i) + \max\{t - d_j, 0\}w_j - \pi_j$
```

Based on these two versions of parallel dynamic programming, we implemented two exact parallel b&p algorithms, which are explained below:

- *Strict Parallel Branch-and-Price (SPBP):* Use *s-pdp* to solve the pricing problem during column generation (line 3 of Algorithm 2).

- *Hybrid Parallel Branch-and-Price (HPBP):* Use *l-dpd* to solve the pricing problem during column generation (line 3 of Algorithm 2). If more than one thread is used and when there is no more column with negative reduced cost according to *l-dpd*, permanently switch to *s-pdp* to solve the pricing problem during column generation (line 3 of Algorithm 2).

Since *s-pdp* and *l-pdp* coincide when using a single OpenMP thread, both SPBP and HPBP coincide with the serial b&p algorithm from Section 4 when executed on a single OpenMP thread (serial execution).

6. Computational Experiments

In this section, we specify the design of our experiments and discuss the results and findings from their execution. First, Section 6.1 describes the computational environment in which the experiments were conducted and introduces the benchmark instances used to evaluate our algorithms. Next, an initial set of experiments to analyze our serial algorithm and its features is discussed in Section 6.2. An important goal of this section is to identify the best performing combination of features in our algorithms, for which the computationally expensive parallel execution will be evaluated next. The following Section 6.3.1 details the in-depth experiments conducted to evaluate the performance benefits obtained from our two parallelization approaches. Finally, we discuss and compare the scalability of our two parallelization strategies in 6.3.2, analyzing which and how the different facets of our algorithm affect it.

6.1. HPC Environment and Benchmark Instances

The experiments are conducted on the Linux-based HPC cluster *Noctua 2* of the *Paderborn Center for Parallel Computing* (PC^2) at Paderborn University. We used a two-socket AMD Milan 7763 shared-memory system with 64 cores per socket, a clock speed of 2.45 GHz per core, and a total of 256 GB main memory. The upper bound on the running time (wall time) for a single compute task executed on the cluster is 21 days (or 504 hours). The b&p algorithms are coded in `C++` and compiled by the `g++` (v12.2.0) compiler with optimization flag

-O3. To solve the restricted linear programs during column generation (line 2 of Algorithm 2), we use the *Gurobi* 10.0.3 API. The parallelization on shared memory is based on OpenMP 4.5.

Three established benchmark sets from the literature are used to evaluate the proposed parallel algorithms: First, the algorithms are tested on the 120 benchmark instances from Cicirello (2003) and Cicirello (2009)[4], which are of problem type $1 \mid ST_{sd} \mid \sum w_j T_j$. Each instance consists of 60 jobs with processing times p_j and weights w_j generated from the integer uniform distribution between 50 and 150, and 0 and 10, respectively. The due dates d_j and setup times s_{ij} are characterized by the three parameters τ, R and η, which define the tightness and the range of the due dates, as well as the size of the average setup time with respect to the size of the average processing time. Cicirello (2003) and Cicirello (2009) created twelve combinations of parameter settings and generated 10 instances for each combination. In the following, we refer to the benchmark set of Cicirello (2003) and Cicirello (2009) as the *Cicirello set*.

Second and third, we apply our algorithms to the benchmark sets from Rubin and Ragatz (1995) and Gagné et al. (2002).[5]These benchmark instances are of type $1 \mid ST_{sd} \mid \sum T_j$. Our algorithms are applicable to this kind of problem by defining the weights for all jobs as one. Rubin and Ragatz (1995)'s set comprises 32 instances with 15, 25, 35 and 45 jobs while Gagné et al. (2002)'s set comprises 32 instances with 55, 65, 75 and 85 jobs. In both sets, processing times are normally distributed with a mean of 100, and setup times are uniformly distributed between 0 and 20. Furthermore, the instances are characterized by the parameters processing time variance, tardiness factor, and due dates range, which build eight possible parameter settings. We refer to the benchmark sets of Rubin and Ragatz (1995) and Gagné et al. (2002) as the *Rubin set* and the *Gagné set*, respectively.

[4]`https://www.cicirello.org/datasets/wtsds/` (last accessed 07/15/24)

[5]`http://www.uqac.ca/portfolio/carolinegagne/recherche/ordonnancement_n_travaux/` (last accessed 07/15/24)

6.2. Serial Performance

In this section, we evaluate the performance of our serial algorithms, in particular, we investigate which benefits regarding running times are gained from the modified branching strategy and the primal heuristic introduced in Sections 4.1 and 4.3. For this series of experiments, we consider all four combinations of the two features in the algorithm. The versions examined are summarized in Table 1.

Table 1: Tested versions of our serial algorithms

Acronym	Primal Heuristic	Branching Strategy
SPBP-PH	✓	modified
SPBP		modified
MF-SPBP-PH	✓	most-fractional
MF-SPBP		most-fractional

$^{\text{SPBP}}$ Strict Parallel Branch-and-Price

$^{\text{PH}}$ Primal Heuristic

$^{\text{MF}}$ Most-Fractional Branching

We apply the algorithms to all instances from the three instance sets and set a time limit of one hour. Table 2 shows the number of instances solved within the time limit for SPBP-PH, SPBP. MF-SPBP-PH and MF-SPBP. Detailed running times for all instances that were solved within one hour by at least one version of the algorithm are given in Appendix A. From the Rubin set, both versions using the modified branching strategy, SPBP-PH and SPBP, solved 29 instances, while MF-SPBP-PH and MF-SPBP, using *most-fractional branching*, both solved 27 of the instances. For the Gagné set, all four versions of the algorithm solved the same 14 instances within the time limit. As for the Cicirello instances, the differences are more distinctive, with 53, 45, 27, and 25 instances solved by the algorithms, respectively.

From these numbers, we can see that each of the two features individually

Table 2: Solved instances within the first hour for SPBP-PH, SPBP, MF-SPBP-PH and MF-SPBP

#solved	SPBP-PH	SPBP	MF-SPBP-PH	MF-SPBP	(total)
Rubin	29	29	27	27	(32)
Gagné	14	14	14	14	(32)
Cicirello	53	45	27	25	(120)
total	96	88	69	66	(184)

has a positive effect on the total number of instances solved within the time limit, and thus both features enhance the performance of the serial algorithm. In addition, the branching strategy affects the benefits gained from the primal heuristic, with 8 more instances solved with the modified strategy compared to two instance when *most-fractional branching* is applied. Since both features are from the serial portion of our algorithm, all further experiments will be performed with both the modified branching strategy and the primal heuristic.

Before proceeding to the parallelization of the two identified algorithms, another parameter we analyzed is the number of columns per iteration (i.e., per call of the dynamic programming algorithm) to be added to the *restricted LP*. In Line 9 of Algorithm 2, only a single column with minimal reduced costs is selected, whereas in fact often more than one *negative reduced cost schedule* may be computed in a single call to the DP algorithm. Especially in the early stages of the b&b procedure, this can cause the same column to be generated again in a subsequent iteration if it was not added to the *restricted LP* because another column was generated at the time with even lower reduced costs. Preliminary tests with up to ten columns per iteration showed that (up to) four columns per iteration resulted in a robust trade-off between saving calls to the DP algorithm without inflating the size of the *restricted LP*. This number was used in all experiments, including those discussed above.

6.3. Parallel Performance

The focus of this section is on the parallelization of our algorithms; i.e., we study the impact of the number of threads used during column generation for both the strict and hybrid approaches. In Section 6.3.1, we discuss the results of the experiments in terms of solved instances, before analyzing our algorithms in terms of speedup; i.e., the benefit gained from using additional computational resources, in Section 6.3.2.

Due to the results discussed in Section 6.2, we consider the algorithm using both the modified branching strategy and the primal heuristic, that is, algorithms SPBP-PH and HPBP-PH. The former algorithm uses the strict strategy in dynamic programming, where threads wait for missing values to become accessible before continuing with the next calculation. The latter uses the hybrid strategy, where missing values are ignored for as long as possible before switching to the strict mode. Both algorithms are executed on all instances of the three benchmark sets, using $th \in \{1, 2, 4, 8, 16, 32, 64, 128\}$ threads for up to 504 hours (21 days).

To compare our results with other exact methods, we use as reference data the algorithm used of Tanaka and Araki (2013). Another exact algorithm worth mentioning is from a study by Sewell et al. (2012). The authors addressed the problem without tardiness weights in the objective function. Using their Branch-and-Bound-and-Remember (BB&R) algorithm, they were able to solve most instances of the Rubin set and some of the Gagné set within 30 minutes. Since they did not consider weights, the authors also did not apply their algorithm to the Cicirello set, and the reported results were not superior to those of the former study, we will only report results from the former study for comparison. Since the ~~their~~aforementioned study is the only one in the literature to report (provably) optimal solutions for most larger instances (up to 85 jobs), their algorithm can be considered the state-of-the-art in exact solution methods for WTSDS. Due to improvements in hardware performance since the release of their study, we re-ran their program with the same settings as in their study, but on the same hardware as we used to run our experiments (remark: we did

Table 3: Solved instances over time using 1 and 64 threads in HPBP-PH

Set	#instances	#threads	Timelimit (h)				(Tanaka*)
			1	12	48	504	(336 h)
Rubin	32	1	29	29	31	32	(32)
		64	29	31	32	32	
Gagné	32	1	14	21	26	30	(29)
		64	21	27	30	31	
Cicirello	120	1	53	99	107	116	(118)
		64	97	111	117	117	

HPBP-PH Hybrid Parallel Branch-and-Price with Primal Heuristic

not use the preprocessed instances Nos. 1–40 of the Cicirello set to run our two algorithms). Although, we could not run the very long jobs (30 days and longer), since all running times are down by up to more than 50% (with very few exceptions), we use these results as reference in all following tables (in this Section, marked with a “*”). For details on the re-enacted experiments and the individual running times, see Appendix C.

6.3.1. Impact of Parallelization

The performance results for our two algorithms HPBP-PH and SPBP-PH in terms of solved instances are presented in Tables 3 and 4, respectively. For each set of benchmark instances, the tables show the total number of instances in the set and the number of instances that our algorithms were able to solve within one hour, twelve hours, two days (48 hours) and 21 days (504 hours). These numbers are shown for the algorithms running in serial mode (one thread) and using 64 parallel threads. The last column in both tables shows the number of instances solved by the algorithm from Tanaka and Araki (2013) for comparison.[6] We

[6]Note, that for instance 751 from the Gagné set and instance No. 7 from the Cicirello set, HPBP-PH on a single thread took more than 14 days (336 hours) to prove optimality. The

Table 4: Solved instances over time using 1 and 64 threads in SPBP-PH

Set	#instances	#threads	Timelimit (h)				(Tanaka*)
			1	12	48	504	(336 h)
Rubin	32	1	29	29	31	32	(32)
		64	29	31	32	32	
Gagné	32	1	14	21	26	30	(29)
		64	20	26	30	31	
Cicirello	120	1	53	98	107	116	(118)
		64	95	111	116	117	

SPBP-PH Strict Parallel Branch-and-Price with Primal Heuristic

choose the 64-thread version as a representative because this version of the algorithms achieves the best overall performance. The complete data for all numbers of threads considered, as well as detailed running times for all instances (including the state-of-the-art running times from Tanaka and Araki (2013) for comparison) are provided in Appendix A. Since the one-hour runs already showed a massive performance degradation, runs with 128 threads were excluded for all further experiments to reduce the computational quota (just the final execution of our experiments alone consumed about 3 million CPU hours).

Using 64 threads, HPBP-PH solved all 32 instances of the Rubin set within 48 hours, all instances but problem 855 of the 32 instances in the Gagné set, and 117 of the 120 instances in the Cicirello set within 21 days to proven optimality. Absolute results for SPBP-PH are comparable. For a closer comparison of the two versions, another measure of performance is discussed in the following section.

For the remaining three instances from the Cicrello set, Nos. 8, 18, and 24, we used 128 GB of RAM which didn't suffice. Since this limit was reached in

same is true for instance 751 from the Gagné set when SPBP-PH was run on a single thread.

less than a week of computing time, we didn't attempt to run them again, even with the available maximum of 256 GB.

We see that for both algorithms, as the number of threads increases, more instances can be solved as well. While using 64 threads results in the same number of solved instances as using 32 threads, running the algorithms with 128 threads actually shows a performance degradation with 10 fewer solved instances from the Cicirello set within the first hour (87 compared to 97 using 64 threads). As mentioned in Section 5, this effect is to be expected, with the values of setup plus processing time ($s_{ij} + p_j$) averaging at 148 for the Cicirello, 108 for the Gagné and 104 for the Rubin instances. The same effects are reflected in the following speedup analysis.

For the three instance sets, Tanaka and Araki (2013) solved all instances to proven optimality within 34 days, except for the instances designated as 851 and 855 from the Gagné set. We were able to solve instance 851 to proven optimality within seven hours using 64 threads (3.5 days in serial execution), showing that the aforementioned authors had found the optimal solution with an objective value of 360. While we can see that their algorithm still has considerably lower overall running times (especially for the Cicirello instances), this shows how difficult it is to compare two b&b algorithms that use different approaches to compute bounds. As for instance 855, we were not able to prove optimality within 21 days, but found a solution with an objective value of 256 (current lower bound of 253.263), which was also found by Sewell et al. (2012) and Xu et al. (2014b) using a BB&R and hybrid evolutionary algorithm, respectively.

In comparison to Tanaka and Araki (2013) and our exact algorithms, Chen (2019) was able to find the optimal solution for 119 out of the 120 instances from the Cicirello set within 1500 seconds using their *Iterated Population Based VND* (IPBVND) heuristic. Within a time limit of 100 seconds, IPBVND found all 32 optimal solutions from the Rubin set and 27 solutions from the Gagné set. Using a relaxed time limit of 80.000 seconds, IPBVND also found three of the remaining five best known solutions from the latter set. This shows that heuristic methods can be effective in computing excellent solutions that exact methods take much longer to obtain. On the other hand, heuristics rely on the results from exact methods to perform such analyses, and in many cases include randomized features. For example, this is also the case for IPBVND, where the reported solution value is the best one obtained from multiple runs of the algorithm. Although Chen (2019) evaluates the robustness of computed solution values among multiple executions of the algorithm, the fact that a single run can produce suboptimal solutions shows that exact methods are still necessary and cannot be discarded.

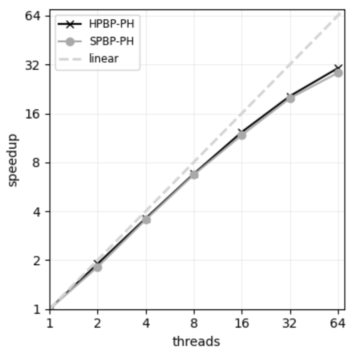
6.3.2. Speedup Analysis

To compare our two parallelization strategies, in this section we analyze the scalability of both approaches. We first introduce an established metric used for this task and discuss the results, before examining the various aspects that lead to the observed effects.

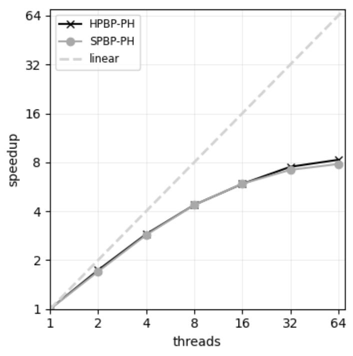
To evaluate the potential from parallelization for algorithms SPBP-PH and HPBP-PH, we use established scalability metrics, namely the parallel speedup and the parallel efficiency (Hager and Wellein, 2010, p.120ff). Parallel speedup on R threads is defined as the ratio of the time to execute the parallel algorithm on one thread to the time to execute the parallel algorithm on R threads, which is called a *relative speedup* (Barr and Hickman, 1993). Parallel efficiency on R threads is defined as the parallel speedup on R threads normalized to the number of threads R.

The median parallel speedups for the three benchmark sets are shown in Figure 1. The left subfigures (a), (c) and (e) show the speedups of dynamic programming (purely parallel part), while the right subfigures (b), (d) and (f) show the overall speedups of the parallel b&p algorithms. The dotted lines represent the so-called *linear speedup*; i.e., a speedup of R on R threads, which corresponds to an efficiency of 100%. Appendix B contains figures for the corresponding parallel efficiencies, as well as the underlying median and average parallel speedup data and their respective standard deviations, also visualized as box plots.

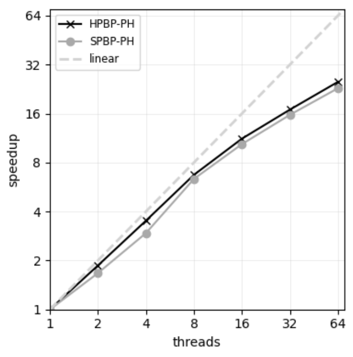
Figure 1: Median parallel speedups



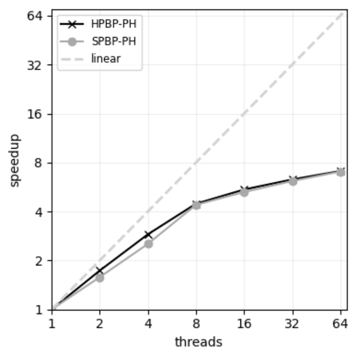
(a) Parallel speedup DP (Cicirello)



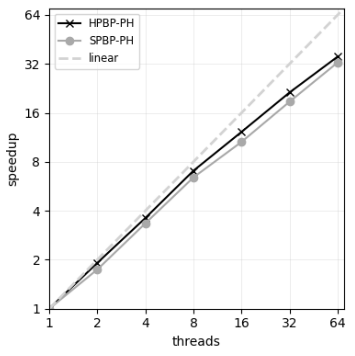
(b) Overall parallel speedup (Cicirello)



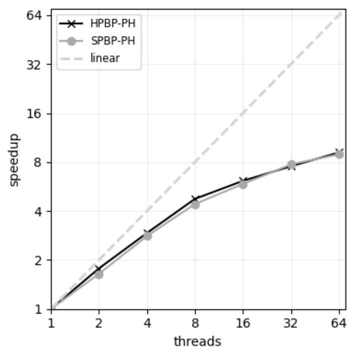
(c) Parallel speedup DP (Rubin)



(d) Overall parallel speedup (Rubin)



(e) Parallel speedup DP (Gagné)



(f) Overall parallel speedup (Gagné)

As can be seen in Figures 1 (a), (c) and (e), the median parallel speedup of the dynamic programming part in HPBP-PH is almost linear for all benchmark sets up to 16 threads. For 32 and 64 threads, the speedup still increases but not nearly as linear as before. Although the speedup values for the dynamic programming parallelization of SPBP-PH are similar to those of HPBP-PH, they are consistently lower. This is due to the high number of calls to the expensive *flush* directive and the time spent waiting for shared variable values to become available. As shown in a corresponding table in Appendix B, for our three benchmark sets, the waiting condition is entered on average 76, 182 and 107 times more often in SPBP-PH than in HPBP-PH, respectively. On the other hand, if enough threads are used, too many values will still be missing when accessed, and HPBP-PH will quickly switch to *s-pdp* mode when no more negative reduced cost columns are found. For the three instance sets considered, this is not yet the case for 128 threads, but the increase in running times for almost all instances compared to 32 and 64 threads clearly indicates that this is the limit for the scalability of our parallelization. The observed running times suggest that peak performance is achieved in the range of 16–32 threads for most instances from the Rubin and Gagné sets, and 32–64 threads for the Cicirello set.

The overall parallel speedups shown in Figures 1 (b), (d) and (f) are significantly lower than the parallel speedups of the dynamic programming part in both SPBP-PH and HPBP-PH. This is due to a non-vanishing serial part, which comprises anything but dynamic programming, and in particular the solution of restricted LPs by Gurobi during column generation (Algorithm 2). Note that using Amdahl's law to provide upper bounds on possible speedups based on the fraction of the serial part (Amdahl, 1967) is not meaningful for our algorithms, since we have observed that the shares in total running time of both the serial and the parallel parts are not constant when different numbers of threads are used.[7]For the Gagné and Rubin instances, the curve of the overall speedup for

[7]The dynamic programming algorithm (i.e., the parallel part of the b&p algorithms) is

both versions becomes almost stagnant using 64 threads (see Figures 1 (d) and (f)); i.e., using more computing resources not only brings no benefit, but may even become detrimental.

Further, we observe that, for the Rubin and Gagné sets, the speedups vary only slightly over all instances considered when 16 or fewer threads are used, while for HPBP-PH, the numbers for the Cicirello set can vary significantly when 16 or more threads are used. The difference in this case can also be seen in a much higher average speedup compared to the median (however, this indicates, that outliers are mostly positive, even though they are the product of coincidentally generating good schedules in a few cases). With the exception of the Cicirello set running HPBP-PH with 8 or more threads, the observed standard deviations are consistently below 1.14 even for 16 threads. We see a different picture for 32 and 64 threads. Here, we even observe a standard deviation of up to 41.33 from the average (overall) speedup of 12.57 for HPBP-PH applied to the Cicirello set, which is considerably high. This also indicates that the scalability of the algorithm is highly dependent on the solved instance, and using too many threads leads to unpredictable behavior (see Footnote 7). These effects are noticeably stronger for HPBP-PH compared to SPBP-PH. The detailed numbers can also be derived from Appendix B. Also, for SPBP-PH, when the number of threads is held constant, the average and median speedup values are similar for the Gagné and Cicirello instances, and slightly lower for the Rubin set, which consists of small instances (45 jobs and fewer). For instances with fewer jobs, threads guickly finish computing all entries from the timeslots in the DP matrix to which they were initially assigned. Thus, during dynamic programming, new timeslots are assigned to threads more frequently to

executed several times during the b&p algorithms. In addition, it cannot be guaranteed that the same set of columns is generated by the b&p algorithms if different numbers of threads are used. Therefore, the serial part of the b&p algorithms (i.e., everything except dynamic programming) is also affected when, for example, more/less columns are generated on a different number of threads.

process the corresponding entries, so the overhead of parallelization is greater, which affects the measured speedup. Regarding HPBP-PH, similar values can be observed across the three sets for up to 8 threads, but variation increases for the Cicirello set. when 16 or more parallel threads are used.

In summary, the application of our two proposed parallel b&p algorithms (HPBP-PH and SPBP-PH) yields substantial speedups that are robust over all sets of benchmark instances up to an instance-specific number of threads. The observed speedup values of the dynamic programming algorithms are similar when the number of threads used is below the limit mentioned above, although they are consistently larger for HPBP-PH compared to SPBP-PH. When using more threads than the aforementioned limit of about 32, the speedups for SPBP-PH are significantly lower compared to HPBP-PH when applied to larger instances, but the standard deviation also increases for the latter version of the algorithm. Since these speedups (for dynamic programming) are reflected in the observed overall speedups (and outliers are mostly on the positive side), HPBP-PH is superior to SPBP-PH with respect to all benchmark sets tested.

7. Conclusion

In this study, we introduce, implement, and evaluate two exact parallel b&p algorithms (refered to as HPBP-PH and SPBP-PH) for the $1 \mid ST_{sd} \mid \sum w_j T_j$ scheduling problem, developed from a serial b&p algorithm we adopted from Lopes and de Carvalho (2007) with a modified branching strategy and an added primal heuristic. For both algorithms, we applied a fine-grained intra-algorithm parallelization strategy by executing the dynamic programming part, which is used to solve the pricing problem during column generation, on up to 128 parallel threads. Before conducting extensive experiments regarding the parallelized part of our algorithms, we verified that the branching strategy and the primal heuristic, both designed to improve the performance of the serial part of the algorithm, allowed us to solve additional instances even within tight time limits. Our computational experiments on established benchmark instances from

Cicirello (2003), Gagné et al. (2002) and Rubin and Ragatz (1995) allowed us to identify an instance-specific limit of threads for which both parallel algorithms achieve substantial speedups with very little variation between instances within the aforementioned benchmark sets. For experiments where this limit is not exceeded, we observe an almost linear speedup for the dynamic programming algorithm in both versions of the algorithm, while the corresponding speedup in SPBP-PH is slightly lower. Also, speedups for SPBP-PH degraded noticeably faster when the number of threads used exceeded the limit. This results in a higher parallelization potential for HPBP-PH compared to SPBP-PH, and thus the former outperforms the latter on all of our benchmark sets tested.

There are limitations to our approach that can be addressed in future work. First, the fine-grained intra-algorithm strategy of our parallelization efforts allows parallelizing the processing of individual nodes of the b&p tree, but still requires sequential processing of the nodes. Parallelizing this processing through coarse-grained intra-algorithm parallelization would allow exploiting this additional speedup potential. When both parallelization strategies are applied jointly, a hybrid parallelization strategy emerges. One possible design of such a hybrid approach is to solve different nodes of the b&p tree simultaneously on different processes, while multiple threads on each process are used to solve the respective b&p node in parallel (e.g., by our parallel dynamic programming algorithms). Second, the observed lower bounds develop more slowly compared to the upper bounds during the execution of the algorithms for most instances. Some acceleration techniques could be implemented and tested, such as dominance rules. While we discarded this idea due to the massive impact on the running time for each call to the dynamic programming algorithm, there may be a strategy that is worth the overhead. Third, the primal heuristic used in our algorithm accounts for less than 0.1% of the total running time. Using a more sophisticated procedure could lead to a higher frequency of finding improving upper bounds, even when a tight time limit is invoked.

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Appendix A. Detailed Running times

This section of the appendix provides detailed results of the experiments discussed in Section 6. Appendix A.1 contains data for HPBP-PH. For the parallelization experiments in 6.3.1, Table A.1 shows a summary of the number of solved instances over increasing time limits for different numbers of threads, while Tables A.2, A.4 and A.6 give the detailed running times of the algorithm using up to 64 threads for the instances from the Rubin, Gagné, and Cicirello sets, respectively. For each instance ("ins" column), the second and third columns of the tables show the objective value found by Tanaka and Araki (2013) and the computation time to find that value. The fourth column shows the objective value found by our algorithm, followed by seven columns (the "t_i" columns) with the computation times needed to solve the instance when using up to 64 threads. Whenever the time limit was exceeded and optimality was not proved, it is indicated by a dash.

For the sake of completeness, Tables A.3, A.5, and A.7 list the instances and running times of each benchmark set using 128 threads, only if they were solved within one hour. As discussed in 6, no further experiments were run with 128 threads due to a significant performance degradation in this setting.

Appendix A.2 contains the same tables as Appendix A.1, only for algorithm SPBP-PH.

Finally, Appendix A.3 shows additional statistics for the experiments considering the different serial versions of the algorithms compared in Section 6.2. Tables A.15 – A.17 summarize statistics for each benchmark set, such as the number of instances solved within one hour for all versions considered, with additional information on the averages of the number of b&b-nodes used, the time taken to solve a single b&b-node, the number of columns generated, and the number of columns generated per node. The specific values used to calculate the above averages are given in the following tables A.18 – A.20. For each instance solved to optimality within the time limit by at least one serial version of the algorithm, they show the running times, the number of b&b nodes, and the number of columns generated for all versions.

Appendix A.1. Hybrid Parallel Branch-and-Price with primal heuristic

Table A.1: Solved instances over time in HPBP-PH

Set	#instances	#threads	Timelimit (h)			
			1	12	48	504
Rubin	32	1	29	29	31	32
		2	29	31	31	32
		4	29	31	32	32
		8	29	31	32	32
		16	29	31	32	32
		32	29	31	32	32
		64	29	31	32	32
Gagné	32	1	14	21	26	30
		2	15	22	27	31
		4	16	23	27	31
		8	19	24	27	31
		16	20	23	28	31
		32	20	26	30	31
		64	21	27	30	31
Cicirello	120	1	53	99	107	116
		2	65	103	110	117
		4	73	105	111	117
		8	84	109	111	117
		16	92	111	115	117
		32	93	111	116	117
		64	97	111	117	117

Table A.2: Running times HPBP-PH over threads for Rubin instances

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
401	90	0.31	90	0.76	0.49	0.33	3.07	0.27	0.38	0.23
402	0	0.00	0	0.31	0.20	0.13	0.58	0.20	0.13	0.26
403	3,418	0.53	3,418	0.46	0.26	0.20	1.69	0.16	0.14	0.28
404	1,067	0.44	1,067	0.42	0.36	0.60	0.19	0.19	0.85	1.73
405	0	0.00	0	0.07	0.06	0.05	0.08	0.05	0.11	0.06
406	0	0.00	0	0.31	0.19	0.12	0.32	0.10	0.21	0.10
407	1,861	0.48	1,861	5.49	3.61	3.16	1.90	1.74	1.53	1.10
408	5,660	0.85	5,660	0.56	0.35	0.33	0.37	0.32	0.13	0.18
501	261	2.09	261	13.56	8.00	5.09	4.79	2.78	2.23	1.46
502	0	0.01	0	0.01	0.02	0.02	0.04	0.04	0.03	0.06
503	3,497	3.04	3,497	15.09	9.09	6.61	4.05	3.01	2.61	2.99
504	0	0.03	0	0.02	0.01	0.01	0.02	0.02	0.06	0.05
505	0	0.02	0	0.79	0.46	0.30	0.46	0.64	0.20	0.39

Table A.2 (cont. - Running times HPBP-PH Rubin)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
506	0	0.02	0	4.94	3.21	2.39	1.59	1.16	1.10	0.99
507	7,225	4.22	7,225	13.24	8.46	7.16	4.05	2.93	2.42	1.43
508	1,915	3.84	1,915	4.18	2.52	2.25	1.19	0.85	0.70	0.96
601	12	7.31	12	59.46	30.79	23.80	12.63	9.74	7.27	5.52
602	0	0.05	0	19.25	10.77	11.69	5.20	3.76	3.19	3.07
603	17,587	18.37	17,587	326.75	181.07	109.03	65.84	52.11	44.34	34.88
604	19,092	25.73	19,092	276.97	159.85	89.82	59.08	47.27	40.14	39.03
605	228	13.57	228	63,921.60	40,444.90	33,674.60	19,114.10	13,954.80	18,682.40	12,130.40
606	0	0.05	0	10.16	6.11	8.17	2.20	1.81	1.82	3.60
607	12,969	17.60	12,969	216.74	121.80	74.87	48.53	39.69	33.48	30.40
608	4,732	32.92	4,732	166.86	97.24	57.37	38.30	31.15	26.50	22.03
701	97	49.55	97	59,237.60	35,380.40	24,331.40	14,977.20	10,276.90	11,626.20	10,374.20
702	0	0.09	0	0.03	0.02	0.01	0.02	0.02	0.02	0.04
703	26,506	52.36	26,506	2,232.51	1,246.19	700.21	409.36	409.46	339.51	279.18
704	15,206	71.40	15,206	401.06	236.39	141.19	90.47	70.20	62.52	76.28

Table A.2 (cont. - Running times HPBP-PH Rubin)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
705	200	599.48	200	276,520.00	154,790.00	97,719.10	74,613.50	60,567.70	55,560.40	40,603.30
706	0	0.10	0	0.02	0.02	0.02	0.01	0.02	0.07	0.04
707	23,789	52.32	23,789	2,718.81	1,543.32	912.34	565.42	444.44	503.30	307.31
708	22,807	99.77	22,807	537.58	314.38	214.78	120.08	105.89	92.32	76.78

Table A.3: Running times HPBP-PH with 128 threads for Rubin instances solved within 1 hour (29 of 32)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
401	0.61	402	0.20	403	0.21	404	0.88	405	0.12	406	0.27
407	3.61	408	0.32	501	3.43	502	0.05	503	4.43	504	0.05
505	0.28	506	2.13	507	3.64	508	1.56	601	7.57	602	2.21
603	70.18	604	66.93	606	5.70	607	57.20	608	53.22	702	0.03
703	383.42	704	93.65	706	0.05	707	529.21	708	153.24		

Table A.4: Running times HPBP-PH over threads for Gagné instances

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
551	183	116.23	183	581.69	322.88	180.51	104.69	103.53	62.97	72.54
552	0	0.17	0	0.02	0.02	0.07	0.01	0.04	0.15	0.03
553	40,498	112.04	40,498	4,489.52	2,528.38	1,559.80	961.26	750.93	462.75	396.06
554	14,653	277.87	14,653	537.38	309.23	207.74	131.28	83.53	68.37	53.77
555	0	0.27	0	366.56	211.41	122.63	72.71	59.62	49.79	35.56
556	0	0.18	0	0.04	0.02	0.02	0.02	0.03	0.07	0.03
557	35,813	140.91	35,813	81,157.90	48,404.80	28,310.70	20,094.60	13,835.20	11,550.10	10,318.10
558	19,871	228.58	19,871	2,008.25	1,067.57	1,229.77	421.65	332.80	279.99	333.73
651	247	2,298.27	247	7,894.18	4,260.64	2,369.51	1,340.28	903.11	702.39	785.19
652	0	0.25	0	0.03	0.02	0.03	0.02	0.03	0.07	0.03
653	57,500	262.80	57,500	56,300.40	35,414.90	19,252.00	12,887.30	11,855.50	10,637.20	7,110.25
654	34,301	381.56	34,301	13,822.30	7,794.42	4,637.62	2,919.35	2,152.15	1,792.97	1,690.33
655	0	355.72	0	1,546.44	840.77	493.00	264.19	188.54	137.11	98.23

Table A.4 (cont. - Running times HPBP-PH Gagné)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
656	0	0.29	0	0.03	0.02	0.02	0.04	0.04	0.05	0.05
657	54,895	253.71	54,895	20,946.10	11,325.60	6,368.04	4,861.22	2,695.80	2,043.43	2,184.69
658	27,114	466.63	27,114	14,618.20	8,348.02	5,070.36	3,433.72	2,262.53	2,250.13	2,183.71
751	225	–	225	1,734,710.00	799,259.00	451,773.00	260,670.00	199,572.00	140,400.00	125,786.00
752	0	0.44	0	0.03	0.03	0.02	0.03	0.03	0.04	0.06
753	77,544	565.95	77,544	38,272.60	20,265.80	13,039.50	7,209.60	6,459.31	5,973.60	3,584.04
754	35,200	887.85	35,200	161,347.00	96,021.10	59,851.40	39,158.40	34,203.60	33,641.60	22,041.10
755	0	0.41	0	80.58	42.80	46.41	12.76	8.60	3.51	4.05
756	0	0.52	0	0.02	0.02	0.03	0.03	0.04	0.06	0.07
757	59,635	741.33	59,635	992,190.00	548,745.00	316,801.00	205,914.00	171,721.00	138,385.00	142,180.00
758	38,339	987.59	38,339	13,087.00	7,384.84	4,563.21	3,036.63	2,265.90	1,911.95	1,631.00
851	363	–	**360**	293,887.00	170,756.00	67,051.40	54,795.80	46,168.30	34,727.00	20,216.70
852	0	0.61	0	0.03	0.03	0.05	0.03	0.06	0.03	0.07
853	97,497	2,219.22	97,497	150,326.00	112,859.00	55,120.20	78,984.20	37,061.30	34,812.60	26,296.30
854	79,042	1,177.04	79,042	170,598.00	98,652.60	59,548.70	37,920.80	27,911.20	23,744.10	19,536.00

Table A.4 (cont. - Running times HPBP-PH Gagné)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
855	260	–	**256**	—	—	—	—	—	—	—
856	0	0.63	0	0.02	0.03	0.04	0.02	0.07	0.07	0.12
857	87,011	4,594.41	87,011	953,237.00	500,272.00	275,904.00	177,832.00	139,693.00	109,228.00	96,656.30
858	74,739	1,817.68	74,739	—	1,297,410.00	759,494.00	486,510.00	508,457.00	396,232.00	386,687.00

Table A.5: Running times HPBP-PH with 128 threads for Gagné instances solved within 1 hour (18 of 32)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
551	157.78	552	0.06	553	802.33	554	126.40	555	95.41	556	0.06
558	698.23	652	0.10	654	1,860.93	655	289.42	656	0.07	657	2,674.07
752	0.12	755	23.73	756	0.13	758	2,201.67	852	0.13	856	0.13

Table A.6: Running times HPBP-PH over threads for Cicirello instances

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
1	453	93.58	453	738,050.00	418,241.00	253,589.00	175,362.00	165,861.00	166,537.00	134,565.00
2	4,794	1,474.67	4,794	46,882.30	27,721.00	17,304.70	9,753.96	7,223.23	4,892.97	10,447.30
3	1,390	589.07	1,390	162,174.00	96,175.60	54,025.70	32,396.90	24,020.50	19,340.60	17,311.10
4	5,866	129.80	5,866	15,539.40	9,158.78	6,497.61	4,052.13	3,376.39	3,014.43	3,478.35
5	4,054	2,350.37	4,054	186,962.00	113,541.00	70,061.60	39,090.40	29,617.60	23,942.80	22,598.90
6	6,592	166.86	6,592	7,027.44	4,180.15	2,746.27	2,149.91	1,600.41	1,340.05	1,943.94
7	3,267	3,522.47	3,267	1,235,430.00	685,638.00	376,841.00	227,258.00	155,839.00	159,457.00	119,181.00
8	100	105.54	100	—	—	—	—	—	—	—
9	5,660	124.33	5,660	58,244.40	34,481.50	19,301.90	11,641.50	8,595.91	6,884.01	6,520.25
10	1,740	8,883.74	1,740	674,219.00	389,384.00	248,377.00	162,748.00	157,431.00	131,881.00	129,288.00
11	2,785	42,921.26	2,785	657,229.00	351,074.00	195,740.00	136,593.00	96,718.40	89,086.30	67,863.90
12	0	0.41	0	239.88	144.75	80.13	55.92	34.84	29.15	27.74
13	3,904	8,722.61	3,904	804,494.00	443,670.00	257,297.00	169,091.00	144,871.00	116,698.00	100,173.00

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
14	2,075	3,555.71	2,075	207,036.00	120,214.00	65,288.70	37,976.80	26,492.30	20,102.00	16,601.40
15	724	496.78	724	20,272.50	12,010.30	6,622.83	4,022.50	3,016.53	2,448.73	2,126.71
16	3,285	504.49	3,285	57,173.00	35,182.10	18,470.40	10,460.50	7,395.82	5,836.84	4,729.33
17	0	1,189.80	0	675.48	444.96	212.60	125.09	93.92	71.24	42.03
18	773	–	**767**	—	—	—	—	—	—	—
19	0	3.41	0	12,962.30	8,117.18	4,350.02	2,765.81	2,527.24	89.64	145.83
20	1,757	316.22	1,757	35,709.50	21,574.40	11,607.30	7,434.90	5,604.76	4,709.62	3,248.46
21	0	0.20	0	284.65	206.25	95.74	64.14	47.63	37.09	33.17
22	0	0.23	0	272.59	154.79	91.59	61.03	47.53	35.41	36.87
23	0	0.17	0	0.02	0.02	0.02	0.03	0.04	0.03	0.12
24	**761**	–	782	—	—	—	—	—	—	—
25	0	0.31	0	385.12	216.28	122.49	182.92	79.71	59.74	43.18
26	0	0.23	0	293.13	156.42	94.28	63.66	45.87	37.01	33.47
27	0	0.34	0	563.74	315.37	185.31	125.56	104.73	75.78	75.15
28	0	0.40	0	561.77	312.44	228.90	141.56	81.11	67.75	138.11

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
29	0	0.23	0	291.79	166.80	129.50	79.50	40.90	38.88	37.84
30	0	7.85	0	1,884.77	1,072.74	304.79	245.48	360.31	120.33	88.34
31	0	0.45	0	326.00	182.51	110.70	70.37	44.37	33.33	68.36
32	0	0.48	0	360.44	204.78	115.74	85.51	54.30	106.53	40.41
33	0	0.48	0	424.56	233.91	176.16	80.57	79.14	42.57	41.29
34	0	0.40	0	0.18	0.21	0.22	0.09	0.06	0.04	0.09
35	0	0.47	0	0.16	0.19	0.13	0.08	1.23	0.04	0.08
36	0	0.44	0	343.26	188.19	124.45	72.72	59.07	36.75	30.26
37	0	595.89	0	2,106.18	1,228.23	679.50	441.75	275.39	211.30	3,461.21
38	0	0.38	0	0.18	0.18	0.12	0.10	0.15	0.04	0.04
39	0	0.48	0	370.15	208.21	114.28	106.08	49.59	43.65	34.12
40	0	0.40	0	377.01	211.19	116.31	84.91	48.67	38.32	29.77
41	69,102	32.06	69,102	189.20	105.95	62.07	45.10	30.41	24.82	19.27
42	57,487	45.69	57,487	5,034.76	3,055.21	1,959.84	1,695.58	1,208.39	1,080.91	1,114.24
43	145,310	80.55	145,310	3,196.19	1,778.61	1,070.34	845.56	542.22	457.19	380.69

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
44	35,166	58.20	35,166	2,230.09	1,337.65	898.33	737.66	575.48	521.28	490.85
45	58,935	80.98	58,935	8,437.05	5,508.25	3,077.07	2,280.30	1,771.53	1,457.56	1,234.94
46	34,764	54.08	34,764	11,237.50	7,193.17	4,240.14	3,499.23	2,567.75	2,286.80	2,295.43
47	72,853	61.35	72,853	8,055.50	4,981.94	2,807.94	2,331.52	1,661.37	1,414.07	1,439.80
48	64,612	105.70	64,612	27,606.70	16,597.30	8,896.76	5,258.33	3,938.23	3,195.50	2,984.14
49	77,449	59.38	77,449	1,040.95	620.55	388.81	311.07	234.69	207.59	217.79
50	31,092	59.55	31,092	4,045.10	2,421.64	1,744.35	1,260.79	1,069.90	957.95	908.71
51	49,208	91.84	49,208	9,507.31	5,275.98	3,127.90	1,962.65	1,490.49	1,252.84	1,211.09
52	93,045	117.65	93,045	12,190.00	7,300.46	3,897.61	2,422.85	1,599.22	1,256.31	1,106.22
53	84,841	116.94	84,841	4,997.73	2,769.89	1,631.14	970.62	717.64	571.50	683.33
54	118,809	104.63	118,809	22,261.00	12,330.30	7,691.73	4,441.04	3,268.75	2,662.15	2,603.50
55	64,315	108.07	64,315	400.54	206.92	119.05	67.19	47.43	41.49	31.98
56	74,889	122.09	74,889	32,306.40	17,852.90	10,031.90	5,820.59	4,151.05	3,199.50	2,862.95
57	63,514	97.11	63,514	14,493.10	8,307.46	5,248.61	3,262.54	2,549.75	2,117.78	1,849.47
58	45,322	135.31	45,322	110,454.00	68,034.20	37,564.40	28,286.10	23,133.60	18,533.60	17,144.30

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
59	50,999	92.49	50,999	14,893.40	8,187.56	5,157.30	3,384.20	2,362.15	2,007.60	1,791.37
60	60,765	137.23	60,765	42,527.60	23,444.30	13,851.00	8,584.94	6,486.70	5,410.47	5,097.46
61	75,916	41.42	75,916	1,725.33	933.65	622.60	384.91	338.03	200.58	195.20
62	44,769	37.31	44,769	116,427.00	76,237.50	51,724.80	39,064.90	32,300.50	28,560.20	29,490.60
63	75,317	33.11	75,317	2,923.13	1,625.64	975.13	804.83	508.59	444.08	551.67
64	92,572	39.94	92,572	3,753.18	2,264.02	1,451.79	1,108.59	981.70	1,110.75	1,077.76
65	126,696	41.08	126,696	1,082.44	623.41	395.96	292.13	229.03	197.68	185.59
66	59,685	17.81	59,685	2,576.15	1,504.15	1,513.20	1,010.95	617.18	559.00	1,136.13
67	29,390	26.93	29,390	1,490.87	905.97	572.25	425.74	317.30	277.16	315.53
68	22,120	25.99	22,120	2,187.11	1,784.17	891.40	630.20	531.23	301.93	364.13
69	71,118	45.03	71,118	28,413.90	17,885.90	12,009.30	8,506.78	8,250.05	3,783.55	3,641.40
70	75,102	36.69	75,102	921.91	531.74	339.07	275.20	207.19	173.76	176.53
71	145,007	96.11	145,007	11,206.50	6,735.04	3,477.83	2,179.62	1,489.48	1,173.23	1,076.41
72	43,286	68.28	43,286	17,043.40	9,079.24	5,349.35	3,355.41	2,117.99	1,599.73	1,382.86
73	28,785	91.49	28,785	57,335.70	35,520.40	19,603.20	13,374.10	10,153.90	9,563.56	7,622.22

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
74	29,777	73.14	29,777	20,701.10	12,401.80	7,629.03	4,918.86	3,912.37	3,299.08	3,205.99
75	21,602	77.57	21,602	2,450.75	1,376.08	834.02	613.10	459.10	387.49	351.15
76	53,555	75.75	53,555	19,937.00	11,442.30	6,913.72	4,387.27	3,514.32	2,931.73	2,705.13
77	31,817	84.48	31,817	11,867.70	6,597.99	4,644.14	2,840.16	1,788.79	1,419.43	1,286.52
78	19,462	79.56	19,462	3,451.97	2,270.12	1,115.59	879.09	552.23	454.31	410.29
79	114,999	70.12	114,999	2,165.96	1,222.21	677.12	475.34	331.05	279.01	253.07
80	18,157	76.73	18,157	15,437.50	8,722.04	5,361.19	2,666.83	2,116.22	1,667.86	1,067.12
81	383,485	30.13	383,485	814.12	528.33	327.69	163.59	221.38	146.37	60.88
82	409,479	66.12	409,479	10,963.40	7,216.17	2,655.10	2,962.72	1,573.76	925.90	1,091.57
83	458,752	42.51	458,752	16,303.20	8,614.88	7,929.49	3,397.98	3,085.43	2,628.61	1,733.58
84	329,670	46.59	329,670	6,016.53	4,606.30	2,666.53	1,635.41	1,576.74	1,151.87	1,179.52
85	554,766	74.38	554,766	33,941.70	19,181.60	10,791.50	6,503.43	4,952.05	3,908.13	3,166.27
86	361,417	68.14	361,417	1,151.19	667.94	374.46	247.38	185.95	153.58	138.19
87	398,551	47.96	398,551	77,025.50	51,584.40	13,899.00	28,421.10	21,428.10	990.97	6,282.45
88	433,186	60.54	433,186	16,027.90	9,100.59	6,516.63	3,191.08	2,428.39	1,961.72	2,630.78

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
89	410,092	46.85	410,092	10,594.30	6,282.13	4,190.29	3,229.05	1,645.60	2,546.36	1,221.00
90	401,653	56.97	401,653	24,829.90	14,178.30	7,511.98	6,872.51	5,959.55	5,506.50	3,937.28
91	339,933	68.98	339,933	5,219.29	2,770.79	1,424.70	2,030.58	955.96	1,326.17	1,296.74
92	361,152	103.36	361,152	6,524.69	3,565.67	2,341.25	1,549.56	1,364.40	970.49	960.60
93	403,423	125.52	403,423	9,903.85	5,250.08	8,970.99	2,105.23	849.66	1,223.02	3,472.39
94	332,941	98.30	332,941	—	1,544,020.00	13,493.80	4,846.76	5,709.92	509,442.00	3,231.40
95	516,926	101.20	516,926	11,017.00	6,287.79	4,058.48	2,745.49	2,165.11	1,866.63	995.33
96	455,448	79.50	455,448	1,212.43	682.69	480.06	251.63	189.35	153.76	120.76
97	407,590	92.92	407,590	2,507.43	1,436.89	932.41	532.35	437.81	358.92	249.43
98	520,582	90.23	520,582	1,123.88	768.73	378.60	267.64	155.63	118.82	167.13
99	363,518	106.21	363,518	955.67	518.80	373.85	220.00	150.26	122.40	109.43
100	431,736	80.98	431,736	2,368.53	1,359.36	851.75	687.37	407.81	340.19	471.17
101	352,990	58.17	352,990	2,238.86	1,287.26	932.22	649.28	512.70	448.70	367.86
102	492,572	55.05	492,572	2,639.89	1,792.91	1,329.65	1,070.20	644.65	594.28	577.61
103	378,602	46.02	378,602	2,249.77	2,881.25	1,033.49	889.21	584.65	549.73	477.26

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka		HPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
104	357,963	56.07	357,963	6,127.37	3,612.43	2,653.60	1,832.78	1,534.37	1,345.85	1,150.95
105	450,806	41.61	450,806	3,503.15	2,107.07	1,648.34	898.80	802.73	709.48	830.75
106	454,379	56.18	454,379	3,545.67	2,231.19	1,515.70	1,034.37	805.25	769.93	746.07
107	352,766	41.31	352,766	1,445.77	851.96	690.28	514.57	385.58	352.43	357.19
108	460,793	39.30	460,793	2,147.61	1,271.65	999.40	540.53	345.77	295.77	293.42
109	413,004	54.97	413,004	432,592.00	273,070.00	186,337.00	62,381.50	1,323.62	37,023.60	39,682.00
110	418,769	60.62	418,769	5,894.66	3,588.86	2,798.92	1,372.20	792.82	1,341.13	409.84
111	342,752	101.73	342,752	5,486.56	3,043.86	1,963.67	967.39	709.30	543.10	596.02
112	367,110	114.06	367,110	3,138.09	1,710.08	1,013.18	563.55	410.97	326.76	272.92
113	259,649	107.98	259,649	5,509.58	3,124.46	1,631.67	1,273.87	1,011.24	871.78	969.94
114	463,474	107.27	463,474	3,200.65	1,803.71	1,088.43	711.25	561.96	469.64	520.57
115	456,890	115.58	456,890	10,342.50	5,521.51	3,128.51	2,014.67	1,492.26	1,158.62	1,018.82
116	530,601	107.89	530,601	3,774.90	2,299.98	1,333.40	906.58	712.92	639.94	542.20
117	502,840	117.72	502,840	8,434.68	2,915.53	2,787.13	1,818.96	1,431.50	1,117.85	790.67
118	349,749	62.23	349,749	231,655.00	141,930.00	84,103.60	10,465.40	747.87	16,316.70	529.76

Table A.6 (cont. - Running times HPBP-PH Cicirello)

	Tanaka			HPBP-PH run with #threads						
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
119	573,046	121.78	573,046	8,244.46	4,460.24	2,533.63	1,573.38	1,163.44	772.67	707.33
120	396,183	84.73	396,183	1,247.44	778.44	398.49	247.11	200.97	150.38	146.27

Table A.7: Running times HPBP-PH with 128 threads for Cicirello instances solved within 1 hour (87 of 120)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
6	2,465.02	12	219.80	17	110.32	19	94.57	21	69.75	22	101.36
23	0.10	25	130.79	26	110.56	27	135.25	28	160.17	29	80.02
30	244.66	31	90.65	32	84.06	33	73.25	34	0.24	35	0.33
36	64.63	37	2,350.57	38	0.20	39	71.30	40	57.12	41	36.67
42	2,241.53	43	671.37	44	851.74	45	2,322.29	46	3,131.06	47	2,338.79
49	313.52	50	1,521.66	51	1,535.99	52	2,013.67	53	1,098.85	54	3,392.79
55	646.05	57	3,125.18	59	2,597.17	61	359.02	63	799.99	64	1,377.25
65	341.30	66	363.12	67	597.97	68	612.05	70	282.63	71	1,545.29
72	2,682.94	75	534.11	77	1,793.13	78	797.75	79	348.64	80	1,592.17
81	91.48	82	2,053.63	83	2,555.91	84	1,248.69	86	361.17	89	1,636.34
91	1,246.35	92	2,707.17	93	1,747.27	95	1,706.37	96	217.23	97	485.50
98	197.65	99	180.32	100	601.23	101	742.75	102	953.37	103	1,164.07
104	1,845.02	105	1,015.08	106	1,065.19	107	334.27	108	427.85	109	1,480.51
110	1,651.42	111	1,031.93	112	522.71	113	1,236.49	114	1,350.06	115	1,511.84
116	919.43	117	1,136.17	119	1,052.29	120	226.64				

Table A.8: Solved instances over time in SPBP-PH

Set	#instances	#threads	Timelimit (h)			
			1	12	48	504
		1	29	29	31	32
		2	29	30	31	32
		4	29	31	32	32
Rubin	32	8	29	31	32	32
		16	29	31	32	32
		32	29	31	32	32
		64	29	31	32	32
		1	14	21	26	30
		2	15	22	26	31
		4	16	23	27	31
Gagné	32	8	19	25	27	31
		16	20	24	28	31
		32	20	26	30	31
		64	20	26	30	31
		1	53	98	107	116
		2	61	103	110	117
		4	74	106	110	117
Cicirello	120	8	84	108	111	117
		16	91	110	115	117
		32	94	110	116	117
		64	95	111	116	117

Table A.9: Running times SPBP-PH over threads for Rubin instances

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
401	90	0.31	90	0.87	0.61	0.37	0.60	0.43	0.24	0.18
402	0	0.00	0	0.32	0.29	0.16	0.18	0.10	0.97	0.24
403	3,418	0.53	3,418	0.45	0.36	0.27	0.25	0.14	0.11	0.09
404	1,067	0.44	1,067	0.43	0.42	0.26	0.20	0.13	0.10	0.16
405	0	0.00	0	0.08	0.06	0.11	0.08	0.03	0.12	0.15
406	0	0.00	0	0.29	0.21	0.17	0.16	0.13	0.08	0.19
407	1,861	0.48	1,861	5.47	3.78	2.44	1.87	1.62	1.51	1.70
408	5,660	0.85	5,660	0.55	0.44	0.26	0.16	0.16	0.17	0.23
501	261	2.09	261	13.34	8.93	5.18	3.29	3.18	2.32	2.17
502	0	0.01	0	0.01	0.06	0.01	0.02	0.04	0.02	0.07
503	3,497	3.04	3,497	15.06	10.07	6.00	4.49	4.45	2.62	2.59
504	0	0.03	0	0.01	0.01	0.02	0.01	0.02	0.03	0.07
505	0	0.02	0	0.79	0.58	0.28	0.27	0.18	0.16	0.24

Table A.9 (cont. - Running times SPBP-PH Rubin)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
506	0	0.02	0	4.95	3.25	2.51	1.57	1.17	1.02	1.21
507	7,225	4.22	7,225	12.94	8.48	5.02	3.36	3.09	3.38	3.02
508	1,915	3.84	1,915	4.16	2.68	2.35	1.07	0.85	1.18	0.70
601	12	7.31	12	53.79	34.52	18.71	11.83	9.69	15.93	6.61
602	0	0.05	0	17.36	11.31	8.08	4.25	3.78	12.18	3.12
603	17,587	18.37	17,587	311.55	197.19	130.03	68.58	54.96	48.24	41.69
604	19,092	25.73	19,092	279.03	176.47	116.00	61.61	51.31	46.38	37.12
605	228	13.57	228	65,980.90	42,210.70	37,096.20	20,076.80	15,847.80	12,353.60	11,162.70
606	0	0.05	0	10.00	6.19	4.89	2.37	3.52	1.97	1.80
607	12,969	17.60	12,969	217.88	138.50	94.49	50.29	41.73	44.44	32.11
608	4,732	32.92	4,732	173.77	108.48	63.07	39.51	34.04	28.21	26.15
701	97	49.55	97	60,233.10	37,306.70	22,409.60	16,112.80	11,678.50	8,894.87	7,776.05
702	0	0.09	0	0.01	0.01	0.01	0.06	0.02	0.04	0.09
703	26,506	52.36	26,506	2,162.98	1,377.65	943.06	531.22	456.79	388.37	328.33
704	15,206	71.40	15,206	417.36	266.46	152.12	94.81	75.13	62.41	57.59

Table A.9 (cont. - Running times SPBP-PH Rubin)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
705	200	599.48	200	275,068.00	187,097.00	105,455.00	75,797.30	52,198.50	49,144.60	64,945.50
706	0	0.10	0	0.01	0.01	0.01	0.01	0.02	0.02	0.07
707	23,789	52.32	23,789	2,771.11	1,679.34	1,096.34	596.03	469.54	376.43	339.57
708	22,807	99.77	22,807	556.75	349.40	199.31	125.19	99.00	82.75	79.05

Table A.10: Running times SPBP-PH with 128 threads for Rubin instances solved within 1 hour (29 of 32)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
401	0.44	402	0.22	403	0.32	404	0.44	405	0.29	406	0.27
407	5.02	408	0.53	501	4.28	502	0.10	503	4.75	504	0.10
505	0.62	506	3.22	507	6.88	508	1.48	601	12.39	602	6.17
603	69.35	604	66.55	606	4.87	607	57.29	608	47.93	702	0.10
703	570.83	704	102.57	706	0.10	707	722.90	708	149.49		

Table A.11: Running times SPBP-PH over threads for Gagné instances

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
551	183	116.23	183	594.41	328.63	226.17	113.16	80.42	59.23	49.07
552	0	0.17	0	0.01	0.01	0.01	0.01	0.02	0.03	0.08
553	40,498	112.04	40,498	5,064.50	2,735.47	1,622.80	1,036.18	890.20	645.95	571.87
554	14,653	277.87	14,653	556.77	339.43	202.30	118.86	95.12	71.60	61.72
555	0	0.27	0	389.21	226.76	131.68	75.62	60.35	48.08	43.58
556	0	0.18	0	0.01	0.22	0.02	0.02	0.03	0.04	0.08
557	35,813	140.91	35,813	81,481.50	53,681.70	30,241.20	19,426.20	14,346.00	12,473.80	10,127.80
558	19,871	228.58	19,871	1,851.68	1,141.82	689.12	441.38	344.59	301.87	463.18
651	247	2,298.27	247	8,463.07	4,956.35	2,573.08	1,483.96	1,194.38	744.95	606.51
652	0	0.25	0	0.02	0.22	0.01	0.02	0.11	0.02	0.08
653	57,500	262.80	57,500	55,896.70	36,206.80	22,319.30	14,583.00	10,567.40	10,166.30	8,332.93
654	34,301	381.56	34,301	14,403.10	8,704.86	5,167.69	3,079.38	2,452.78	1,870.45	1,588.65
655	0	355.72	0	1,559.29	925.41	507.17	282.26	222.87	152.73	121.96

Table A.11 (cont. - Running times SPBP-PH Gagné)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
656	0	0.29	0	0.02	0.22	0.07	0.03	0.04	0.02	0.08
657	54,895	253.71	54,895	21,221.10	12,452.80	7,072.64	3,987.54	2,968.20	2,157.51	1,810.51
658	27,114	466.63	27,114	15,103.70	9,172.86	5,351.47	3,581.59	2,974.71	2,444.16	2,340.26
751	225	–	225	1,460,150.00	883,918.00	481,676.00	320,124.00	223,806.00	158,629.00	130,147.00
752	0	0.44	0	0.03	0.20	0.02	0.06	0.08	0.04	0.09
753	77,544	565.95	77,544	38,980.30	24,225.90	14,014.00	9,214.73	6,728.23	5,608.75	4,941.36
754	35,200	887.85	35,200	157,443.00	122,138.00	69,312.60	46,871.40	36,234.70	27,899.20	26,245.70
755	0	0.41	0	87.90	48.26	25.23	13.41	9.31	7.64	8.72
756	0	0.52	0	0.02	0.24	0.02	0.48	0.27	0.03	0.09
757	59,635	741.33	59,635	984,865.00	606,760.00	343,578.00	241,954.00	181,176.00	142,062.00	126,754.00
758	38,339	987.59	38,339	13,718.60	8,082.19	4,650.96	3,195.28	2,312.76	1,855.87	1,651.76
851	363	–	**360**	292,313.00	178,364.00	113,623.00	66,935.70	42,273.60	36,027.60	29,402.40
852	0	0.61	0	0.02	0.25	0.02	0.09	0.04	0.06	0.07
853	97,497	2,219.22	97,497	148,568.00	99,633.40	60,884.60	37,689.40	35,306.40	25,434.40	37,185.70
854	79,042	1,177.04	79,042	171,476.00	111,018.00	72,559.50	41,472.90	32,311.30	26,243.80	19,195.70

Table A.11 (cont. - Running times SPBP-PH Gagné)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
855	260	-	**256**	—	—	—	—	—	—	—
856	0	0.63	0	0.02	0.23	0.02	0.05	0.04	0.03	0.08
857	87,011	4,594.41	87,011	934,332.00	569,395.00	310,244.00	209,474.00	147,052.00	114,258.00	99,223.10
858	74,739	1,817.68	74,739	—	1,682,130.00	943,703.00	675,113.00	623,068.00	515,538.00	366,282.00

Table A.12: Running times SPBP-PH with 128 threads for Gagné instances solved within 1 hour (19 of 32)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
551	114.27	552	0.11	553	905.16	554	117.75	555	120.51	556	0.11
558	518.86	651	1,366.39	652	0.11	654	2,525.70	655	393.90	656	0.10
657	3,610.62	752	0.11	755	15.94	756	0.11	758	2,697.11	852	0.09
856	0.05										

Table A.13: Running times SPBP-PH over threads for Cicirello instances

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
1	453	93.58	453	764,819.00	434,062.00	270,093.00	203,251.00	172,724.00	141,066.00	135,057.00
2	4,794	1,474.67	4,794	46,323.20	30,260.50	17,238.30	10,447.10	7,822.45	6,134.58	5,541.13
3	1,390	589.07	1,390	161,807.00	100,924.00	61,477.60	34,926.80	25,338.70	19,887.40	18,058.70
4	5,866	129.80	5,866	18,020.00	9,794.37	6,321.59	4,188.10	3,523.34	3,326.81	2,936.05
5	4,054	2,350.37	4,054	187,346.00	118,072.00	69,500.50	39,463.40	31,651.10	25,945.20	22,176.40
6	6,592	166.86	6,592	7,407.97	4,366.65	2,994.94	1,987.68	1,569.61	1,393.82	1,288.26
7	3,267	3,522.47	3,267	1,197,540.00	687,397.00	388,091.00	264,245.00	191,244.00	150,127.00	133,418.00
8	100	105.54	100	—	—	—	—	—	—	—
9	5,660	124.33	5,660	58,397.30	36,511.80	21,067.60	13,806.90	9,095.70	7,542.26	9,549.96
10	1,740	8,883.74	1,740	659,084.00	404,800.00	252,465.00	185,270.00	159,510.00	133,952.00	123,176.00
11	2,785	42,921.26	2,785	642,293.00	364,362.00	203,490.00	143,124.00	95,066.20	73,474.30	63,757.20
12	0	0.41	0	254.60	149.37	78.17	45.97	45.12	29.51	25.73
13	3,904	8,722.61	3,904	777,434.00	451,972.00	268,627.00	188,024.00	149,776.00	123,711.00	112,130.00

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
14	2,075	3,555.71	2,075	206,692.00	136,137.00	68,711.20	37,087.30	28,131.20	20,603.70	17,610.80
15	724	496.78	724	21,172.30	12,036.30	6,661.74	5,157.82	3,109.24	2,538.09	2,276.94
16	3,285	504.49	3,285	57,225.20	38,362.20	18,852.10	15,148.80	7,636.27	5,743.31	4,963.54
17	0	1,189.80	0	703.79	406.06	223.21	127.60	104.80	75.80	67.42
18	773	–	**767**	—	—	—	—	—	—	—
19	0	3.41	0	13,813.50	8,215.91	4,502.77	3,545.32	2,448.71	2,064.86	1,869.15
20	1,757	316.22	1,757	37,225.80	21,856.10	13,374.20	8,261.42	5,765.41	4,935.41	5,201.19
21	0	0.20	0	302.59	170.28	97.75	69.86	46.33	44.41	34.67
22	0	0.23	0	291.89	164.07	96.89	65.76	44.60	35.92	33.52
23	0	0.17	0	0.01	0.02	0.01	0.01	0.03	0.05	0.02
24	**761**	–	878	—	—	—	—	—	—	—
25	0	0.31	0	411.21	233.44	131.46	78.01	69.24	70.39	51.61
26	0	0.23	0	308.29	167.75	93.59	56.50	45.90	40.71	33.91
27	0	0.34	0	576.63	333.37	190.33	116.76	93.91	100.45	62.20
28	0	0.40	0	569.51	329.56	183.32	111.35	86.91	69.51	62.66

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
29	0	0.23	0	300.54	171.65	96.60	59.38	48.45	40.06	37.82
30	0	7.85	0	1,869.82	1,146.47	676.72	445.89	452.35	422.75	291.57
31	0	0.45	0	333.14	187.00	98.67	57.00	47.27	37.69	28.88
32	0	0.48	0	366.70	210.39	112.66	65.97	56.83	41.90	36.48
33	0	0.48	0	431.63	243.12	129.59	76.41	55.34	47.29	37.75
34	0	0.40	0	0.18	0.11	0.06	0.07	0.09	0.06	0.06
35	0	0.47	0	0.16	0.11	0.05	0.12	0.08	0.68	0.08
36	0	0.44	0	350.21	195.32	103.71	59.54	45.57	40.63	32.46
37	0	595.89	0	2,137.46	1,223.43	664.85	386.00	325.01	254.98	193.28
38	0	0.38	0	0.16	0.11	0.05	0.06	0.27	0.15	0.38
39	0	0.48	0	388.50	211.10	115.05	65.85	55.07	38.97	33.02
40	0	0.40	0	391.47	214.76	117.45	68.34	52.74	49.54	37.32
41	69,102	32.06	69,102	197.93	115.55	65.94	40.64	31.55	25.81	24.85
42	57,487	45.69	57,487	5,708.65	3,182.31	2,037.80	1,442.22	1,234.92	1,219.92	1,042.50
43	145,310	80.55	145,310	3,148.87	1,903.58	1,121.68	710.50	561.53	466.88	427.15

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
44	35,166	58.20	35,166	2,176.77	1,374.20	933.72	675.59	592.81	516.00	579.17
45	58,935	80.98	58,935	9,115.60	5,673.01	3,231.14	2,215.64	1,855.13	1,774.86	1,535.40
46	34,764	54.08	34,764	11,654.50	7,312.72	4,565.23	3,113.49	2,678.60	2,351.15	2,240.55
47	72,853	61.35	72,853	8,360.93	5,166.25	2,950.39	2,001.94	1,601.44	1,447.86	1,351.23
48	64,612	105.70	64,612	29,552.60	17,667.10	10,637.20	6,203.00	4,192.46	3,330.12	2,885.62
49	77,449	59.38	77,449	1,024.81	687.51	409.37	285.77	239.32	210.97	199.08
50	31,092	59.55	31,092	4,645.45	2,614.82	1,696.37	1,234.75	1,199.18	958.51	997.95
51	49,208	91.84	49,208	10,404.60	6,843.21	3,276.75	2,041.59	1,568.81	1,257.59	1,117.60
52	93,045	117.65	93,045	13,518.60	7,431.77	4,048.03	2,323.00	1,783.39	1,300.97	1,126.97
53	84,841	116.94	84,841	5,696.02	2,969.06	1,661.31	1,025.81	837.19	584.72	519.53
54	118,809	104.63	118,809	24,431.00	13,077.00	8,027.45	5,243.56	3,372.13	2,694.49	2,440.91
55	64,315	108.07	64,315	400.08	238.91	122.19	72.49	49.71	37.02	31.34
56	74,889	122.09	74,889	34,401.30	19,068.50	10,518.20	6,760.98	4,424.26	3,294.30	2,794.30
57	63,514	97.11	63,514	15,633.50	8,818.79	5,728.87	3,389.40	2,621.27	2,173.27	2,006.54
58	45,322	135.31	45,322	110,531.00	70,389.60	40,906.30	27,333.70	22,196.60	17,502.90	16,484.40

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
59	50,999	92.49	50,999	15,788.40	8,742.06	5,530.64	3,211.87	2,498.78	1,987.60	1,783.85
60	60,765	137.23	60,765	39,347.20	24,586.50	14,230.00	11,114.30	7,228.24	5,587.36	6,971.11
61	75,916	41.42	75,916	1,692.68	1,052.40	569.44	343.96	259.24	212.42	183.56
62	44,769	37.31	44,769	117,872.00	79,972.30	40,298.40	39,387.70	32,359.30	29,798.60	27,559.00
63	75,317	33.11	75,317	2,861.67	1,776.20	1,023.50	656.18	515.71	428.89	403.22
64	92,572	39.94	92,572	4,469.63	2,433.89	1,506.70	1,069.80	891.23	939.31	736.52
65	126,696	41.08	126,696	1,068.20	708.34	406.38	280.79	232.44	200.41	203.80
66	59,685	17.81	59,685	2,458.32	1,630.56	1,018.29	733.08	629.57	557.44	559.03
67	29,390	26.93	29,390	1,442.07	947.00	564.25	386.56	329.40	290.74	278.69
68	22,120	25.99	22,120	2,081.19	1,311.07	732.45	536.98	371.00	310.96	334.88
69	71,118	45.03	71,118	29,807.70	18,495.10	12,786.80	11,640.20	8,192.51	4,625.74	8,863.59
70	75,102	36.69	75,102	893.90	596.55	350.36	288.62	202.10	189.42	168.82
71	145,007	96.11	145,007	12,093.30	6,554.68	3,605.16	2,341.81	1,567.67	1,228.40	1,047.04
72	43,286	68.28	43,286	17,621.80	9,595.61	5,271.37	3,057.71	2,223.00	1,655.54	1,400.90
73	28,785	91.49	28,785	57,803.30	36,634.60	22,441.20	16,254.00	10,742.80	8,980.61	8,319.94

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
74	29,777	73.14	29,777	22,520.70	12,914.50	7,690.24	6,755.54	3,982.13	3,340.68	3,074.77
75	21,602	77.57	21,602	2,398.51	1,510.01	801.81	590.24	466.27	393.63	376.10
76	53,555	75.75	53,555	20,577.20	11,774.10	6,993.58	6,133.12	3,560.96	3,040.99	2,728.77
77	31,817	84.48	31,817	12,426.50	6,900.87	3,913.04	2,397.30	1,807.87	1,449.07	1,287.47
78	19,462	79.56	19,462	3,378.59	2,048.43	1,364.95	740.87	574.15	463.65	416.21
79	114,999	70.12	114,999	2,127.41	1,284.75	842.19	460.03	346.41	279.04	245.41
80	18,157	76.73	18,157	15,984.20	9,011.19	5,009.50	3,425.87	2,195.39	1,791.84	1,500.78
81	383,485	30.13	383,485	737.00	497.72	381.36	250.84	221.77	199.36	194.00
82	409,479	66.12	409,479	11,705.20	7,040.22	4,430.60	3,069.58	2,570.08	1,412.15	2,155.13
83	458,752	42.51	458,752	16,796.90	9,873.81	6,323.81	5,364.14	3,159.35	2,772.44	2,508.05
84	329,670	46.59	329,670	6,681.81	3,916.26	2,661.25	1,862.87	1,590.37	1,357.31	1,390.78
85	554,766	74.38	554,766	35,768.70	20,588.00	11,284.10	7,727.14	5,188.70	4,220.73	3,586.68
86	361,417	68.14	361,417	1,151.50	694.49	429.55	279.86	202.62	163.51	144.16
87	398,551	47.96	398,551	77,762.80	53,796.90	40,367.70	27,922.90	22,651.60	20,076.20	19,119.90
88	433,186	60.54	433,186	16,727.80	10,158.40	6,698.28	3,853.34	2,541.64	2,064.79	1,869.23

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
89	410,092	46.85	410,092	10,520.50	7,130.25	5,799.47	3,371.02	2,778.55	2,510.19	2,438.73
90	401,653	56.97	401,653	23,878.20	15,318.40	11,447.40	7,681.06	6,374.23	5,467.48	5,200.92
91	339,933	68.98	339,933	5,497.26	2,937.65	1,791.85	1,220.80	1,023.28	856.59	791.81
92	361,152	103.36	361,152	7,121.18	4,073.99	2,233.87	1,477.01	1,210.78	987.69	912.02
93	403,423	125.52	403,423	10,374.90	5,487.01	3,189.32	1,987.97	1,536.54	1,262.64	1,122.01
94	332,941	98.30	332,941	—	1,417,060.00	931,276.00	679,787.00	619,479.00	631,159.00	574,826.00
95	516,926	101.20	516,926	12,002.90	7,118.30	5,266.95	2,855.92	2,220.89	1,903.90	1,884.07
96	455,448	79.50	455,448	1,221.29	709.73	418.42	281.62	199.89	156.14	141.32
97	407,590	92.92	407,590	2,525.89	1,484.59	884.63	570.12	442.96	366.21	338.23
98	520,582	90.23	520,582	1,113.84	634.85	362.12	232.87	157.18	122.98	110.48
99	363,518	106.21	363,518	922.87	544.00	323.13	239.28	152.94	125.64	118.13
100	431,736	80.98	431,736	2,302.93	1,374.24	813.16	526.28	415.47	346.50	314.13
101	352,990	58.17	352,990	2,122.52	1,353.44	879.36	617.20	526.36	464.54	447.15
102	492,572	55.05	492,572	2,604.45	1,676.38	1,096.58	836.58	663.98	589.48	563.65
103	378,602	46.02	378,602	2,225.61	1,437.94	959.55	756.20	620.95	534.99	506.67

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
104	357,963	56.07	357,963	6,875.76	3,871.10	2,504.74	1,930.98	1,569.30	1,403.32	1,347.61
105	450,806	41.61	450,806	3,456.39	2,178.43	1,400.60	1,056.95	755.82	723.12	709.46
106	454,379	56.18	454,379	3,500.56	2,423.99	1,417.11	1,055.06	828.61	745.94	684.06
107	352,766	41.31	352,766	1,422.14	972.31	609.92	450.59	384.54	345.22	362.39
108	460,793	39.30	460,793	2,143.26	1,323.81	748.53	465.91	358.88	294.13	261.22
109	413,004	54.97	413,004	417,437.00	288,744.00	207,514.00	165,728.00	155,953.00	60,405.10	1,816.81
110	418,769	60.62	418,769	6,599.02	4,224.25	2,671.24	2,046.09	1,783.98	1,624.00	1,453.60
111	342,752	101.73	342,752	6,193.81	3,101.55	1,708.98	1,013.39	746.70	558.42	482.78
112	367,110	114.06	367,110	3,219.53	1,791.80	1,005.38	592.69	433.92	328.10	291.18
113	259,649	107.98	259,649	6,373.45	3,470.44	1,976.84	1,313.70	1,044.19	882.36	839.95
114	463,474	107.27	463,474	3,165.91	1,920.08	1,132.87	735.49	579.46	474.33	448.63
115	456,890	115.58	456,890	10,847.60	5,815.56	3,251.93	1,981.15	1,461.02	1,142.67	1,008.80
116	530,601	107.89	530,601	4,471.09	2,274.17	1,374.35	925.51	731.15	621.91	570.08
117	502,840	117.72	502,840	9,005.66	4,938.52	2,915.08	1,900.23	1,482.87	1,222.67	1,112.71
118	349,749	62.23	349,749	231,111.00	147,701.00	92,251.70	55,174.00	42,446.50	35,042.10	34,875.10

Table A.13 (cont. - Running times SPBP-PH Cicirello)

	Tanaka		SPBP-PH run with #threads							
ins	best	time	our_best	t_1	t_2	t_4	t_8	t_16	t_32	t_64
119	573,046	121.78	573,046	8,851.38	5,157.44	2,665.85	1,640.46	1,213.36	956.61	845.22
120	396,183	84.73	396,183	1,248.65	731.60	414.50	253.81	192.57	153.05	136.06

Table A.14: Running times SPBP-PH with 128 threads for Cicirello instances solved within one hour (84 of 120)

ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128	ins	t_128
6	2,179.07	12	55.79	17	145.75	21	123.07	22	88.92	23	0.10
25	121.74	26	106.49	27	197.25	28	176.19	29	93.40	30	620.97
31	54.02	32	78.64	33	81.74	34	0.38	35	0.20	36	65.87
37	448.37	38	0.44	39	73.87	40	94.31	41	50.46	42	1,749.57
43	806.17	44	1,072.35	45	2,149.37	46	3,109.11	47	2,145.51	49	592.72
50	1,598.03	51	1,673.90	52	2,002.01	53	1,019.30	55	51.15	57	3,303.80
59	2,802.97	61	337.92	63	788.86	64	1,070.95	65	283.15	66	842.37
67	415.32	68	772.89	70	248.10	71	1,675.24	72	2,137.06	75	560.18
77	2,002.14	78	626.54	79	422.99	80	2,532.78	81	260.89	84	2,420.44
86	284.93	88	3,504.64	89	3,408.24	91	848.44	92	1,214.10	93	1,984.76
95	2,361.51	96	200.92	97	532.40	98	270.60	99	200.13	100	414.10
101	912.15	102	738.74	103	1,016.57	104	1,738.29	105	1,527.29	106	926.61
107	427.84	108	640.01	110	2,189.81	111	1,046.99	112	437.69	113	1,035.14
114	548.76	115	1,777.65	116	715.74	117	2,075.06	119	1,296.12	120	204.48

Appendix A.3. Comparison of serial versions of the algorithm

Table A.15: Statistics for the four algorithms on Rubin instances solved within one hour in serial execution (means taken over instances solved by all four versions)

Algorithm	SPHP+PH	SPHP	MF-SPHP-PH	MF-SPHP
#solved	29	29	27	27
#b&b-nodes (∅)	13	20	19	24
time per node (∅)	3.20	3.06	2.96	2.89
#columns (∅)	1905	2084	2586	2714
#columns per node (∅)	51	59	52	56

Table A.16: Statistics for the four algorithms on Gagné instances solved within one hour in serial execution (means taken over instances solved by all four versions)

Algorithm	SPHP+PH	SPHP	MF-SPHP-PH	MF-SPHP
#solved	14	14	14	14
#b&b-nodes (∅)	11	21	13	22
time per node (∅)	18.26	17.92	18.70	18.46
#columns (∅)	2813	3293	3463	3856
#columns per node (∅)	94	100	102	107

Table A.17: Statistics for the four algorithms on Cicirello instances solved within one hour in serial execution (means taken over instances solved by all three versions)

Algorithm	SPHP+PH	SPHP	MF-SPHP-PH	MF-SPHP
#solved	53	45	27	25
#b&b-nodes (∅)	63	70	79	79
time per node (∅)	19.35	22.05	12.12	14.27
#columns (∅)	4524	5343	4806	4916
#columns per node (∅)	98	99	39	10

Table A.18: Comparison of serial execution results for the algorithm using new branching strategy and primal heuristic and using most-fractional branching on Rubin instances

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
401	90	0.31	0.87	3	230	0.81	3	230	0.74	3	220	0.75	3	220
402	0	0.00	0.32	3	98	1.16	15	439	0.94	9	282	1.41	23	659
403	3,418	0.53	0.45	1	116	0.46	1	116	0.48	1	116	0.43	1	116
404	1,067	0.44	0.43	1	129	0.42	1	129	0.47	1	129	0.47	1	129
405	0	0.00	0.08	1	31	0.40	25	227	0.13	1	31	0.40	29	255
406	0	0.00	0.29	1	92	1.41	23	631	0.34	1	92	1.51	23	664
407	1,861	0.48	5.47	27	1,778	5.48	27	1,778	3.91	17	1,260	4.06	17	1,260
408	5,660	0.85	0.55	1	141	0.55	1	141	0.67	1	141	0.70	1	141
501	261	2.09	13.34	7	1,079	13.27	7	1,079	10.46	5	835	10.60	5	835
502	0	0.01	0.01	1	2	0.01	1	2	0.01	1	2	0.01	1	2
503	3,497	3.04	15.06	9	1,132	17.91	11	1,344	17.27	9	1,260	17.33	9	1,260
504	0	0.03	0.01	1	2	0.03	1	2	0.04	1	2	0.03	1	2
505	0	0.02	0.79	1	72	4.74	41	676	0.91	1	72	5.68	37	803
506	0	0.02	4.95	23	540	6.48	35	840	4.94	29	598	6.24	47	1,004
507	7,225	4.22	12.94	7	1,028	12.79	7	1,028	13.47	7	1,075	12.39	7	1,075
508	1,915	3.84	4.16	1	323	4.19	1	323	4.45	1	323	4.01	1	323

Table A.18 (cont. - Serial comparison Rubin)

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
601	12	7.31	53.79	5	1,740	95.18	11	3,231	64.74	7	2,039	61.23	7	2,039
602	0	0.05	17.36	41	982	20.05	61	1,409	21.96	55	1,414	22.21	61	1,617
603	17,587	18.37	311.55	55	8,984	294.75	55	8,984	489.64	89	14,731	482.09	89	14,731
604	19,092	25.73	279.03	61	7,994	264.50	61	7,994	497.28	119	14,689	489.41	119	14,689
606	0	0.05	10.00	25	411	19.44	69	1,135	21.67	61	1,251	22.29	69	1,521
607	12,969	17.60	217.88	23	6,445	204.94	23	6,445	174.67	21	5,283	166.24	21	5,283
608	4,732	32.92	173.77	15	4,652	163.21	15	4,652	119.18	9	3,276	135.41	11	3,961
702	0	0.09	0.01	1	2	0.03	1	2	0.03	1	2	0.01	1	2
703	26,506	52.36	2,162.98	133	32,778	2,121.90	133	32,778	—	—	—	—	—	—
704	15,206	71.40	417.36	21	5,598	399.61	21	5,598	809.69	43	11,223	788.49	43	11,223
706	0	0.10	0.01	1	2	0.01	1	2	0.01	1	2	0.01	1	2
707	23,789	52.32	2,771.11	191	42,262	2,734.51	191	42,262	—	—	—	—	—	—
708	22,807	99.77	556.75	21	7,837	537.53	21	7,837	662.10	31	9,465	662.92	31	9,465

Table A.19: Comparison of serial execution results for the algorithm using new branching strategy with (SPBP-PH)/without (SPBP) primal heuristic and using most-fractional branching (MF-SPBP) on Gagné instances

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
551	183	116.23	594.41	11	5,454	574.69	11	5,454	640.22	13	5,766	638.62	13	5,766
552	0	0.17	0.01	1	2	0.02	1	2	0.05	1	2	0.02	1	2
554	14,653	277.87	556.77	11	4,337	535.48	11	4,337	687.37	15	5,469	691.85	15	5,469
555	0	0.27	389.21	69	6,369	369.24	69	6,369	323.06	81	6,025	319.49	81	6,025
556	0	0.18	0.01	1	2	0.02	1	2	0.02	1	2	0.02	1	2
558	19,871	228.58	1,851.68	21	13,523	1,827.66	21	13,523	1,582.40	19	11,769	1,584.37	19	11,769
652	0	0.25	0.02	1	2	0.02	1	2	0.05	1	2	0.02	1	2
655	0	355.72	1,559.29	19	9,320	1,542.74	19	9,320	3,138.50	31	19,066	2,987.73	31	19,066
656	0	0.29	0.02	1	2	0.04	1	2	0.03	1	2	0.02	1	2
752	0	0.44	0.03	1	2	0.02	1	2	0.02	1	2	0.04	1	2
755	0	0.41	87.90	9	366	604.20	149	7,080	80.05	11	371	569.77	145	5,875
756	0	0.52	0.02	1	2	0.07	1	2	0.02	1	2	0.02	1	2
852	0	0.61	0.02	1	2	0.03	1	2	0.03	1	2	0.02	1	2
856	0	0.63	0.02	1	2	0.05	1	2	0.07	1	2	0.02	1	2

Table A.20: Comparison of serial execution results for the algorithm using new branching strategy with (SPBP-PH)/without (SPBP) primal heuristic and using most-fractional branching (MF-SPBP) on Cicirello instances

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
12	0	0.41	254.60	99	2,934	249.24	105	3,219	242.71	115	3,359	243.28	115	3,359
17	0	1,189.80	703.79	39	4,486	1,108.22	57	8,937	72.24	3	326	—	—	—
19	0	3.41	—	—	—	—	—	—	1,071.94	167	22,001	1,089.16	167	22,001
21	0	0.20	302.59	93	3,885	307.15	107	4,701	378.74	119	6,066	389.45	121	6,371
22	0	0.23	291.89	99	3,667	289.48	111	4,272	389.13	131	6,129	389.44	131	6,129
23	0	0.17	0.01	1	2	0.01	1	2	0.01	1	2	0.01	1	2
25	0	0.31	411.21	97	5,573	578.08	173	12,489	369.05	113	6,158	372.32	113	6,158
26	0	0.23	308.29	105	3,829	306.94	117	4,860	300.64	135	4,815	302.50	135	4,815
27	0	0.34	576.63	67	6,006	603.84	77	6,954	536.34	97	7,513	555.46	99	7,835
28	0	0.40	569.51	77	7,187	566.05	79	7,604	564.59	103	9,270	578.91	103	9,270
29	0	0.23	300.54	111	4,278	288.73	113	4,429	343.84	115	5,871	346.54	115	5,871
30	0	7.85	1,869.82	89	20,204	2,037.94	101	22,877	670.88	73	8,863	700.50	73	8,863
31	0	0.45	333.14	89	2,841	338.91	101	3,395	291.00	129	3,371	293.99	129	3,371
32	0	0.48	366.70	101	4,062	350.35	101	4,062	326.30	123	3,831	329.31	123	3,831
33	0	0.48	431.63	93	3,467	418.37	95	3,542	346.31	131	3,834	339.72	131	3,834
34	0	0.40	0.18	1	2	0.18	1	2	0.20	1	2	0.18	1	2
35	0	0.47	0.16	1	2	0.18	1	2	0.16	1	2	0.16	1	2

Table A.20 (cont. - Serial comparison Cicirello)

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
36	0	0.44	350.21	101	3,134	340.27	107	3,514	339.23	133	4,349	342.12	135	4,458
37	0	595.89	2,137.46	59	15,549	2,134.75	59	15,549	—	—	—	—	—	—
38	0	0.38	0.16	1	2	0.17	1	2	0.18	1	2	0.16	1	2
39	0	0.48	388.50	87	3,165	383.19	91	3,508	323.90	123	4,328	322.84	123	4,328
40	0	0.40	391.47	101	3,820	371.01	101	3,820	300.14	119	3,288	303.18	119	3,288
41	69,102	32.06	197.93	1	1,459	470.63	5	3,496	196.28	1	1,459	292.04	3	2,115
43	145,310	80.55	3,148.87	71	20,705	3,273.58	75	21,942	—	—	—	—	—	—
44	35,166	58.20	2,176.77	33	16,003	3,143.68	55	24,709	3,464.29	51	25,982	—	—	—
49	77,449	59.38	1,024.81	15	6,826	2,547.87	29	18,276	—	—	—	—	—	—
55	64,315	108.07	400.08	5	1,791	533.10	7	2,518	288.85	3	1,314	468.87	5	2,120
61	75,916	41.42	1,692.68	29	12,331	1,911.25	31	14,109	—	—	—	—	—	—
63	75,317	33.11	2,861.67	41	21,118	2,834.91	41	21,118	—	—	—	—	—	—
65	126,696	41.08	1,068.20	13	6,756	1,433.85	19	9,304	—	—	—	—	—	—
66	59,685	17.81	2,458.32	11	16,647	2,437.09	11	16,647	2,899.82	33	21,122	2,893.09	33	21,122
67	29,390	26.93	1,442.07	19	10,260	2,592.62	31	19,721	—	—	—	—	—	—
68	22,120	25.99	2,081.19	39	17,164	2,061.16	39	17,164	—	—	—	—	—	—
70	75,102	36.69	893.90	11	6,105	993.39	13	6,981	810.99	11	5,594	897.91	13	5,914
75	21,602	77.57	2,398.51	25	10,738	3,235.81	35	14,864	—	—	—	—	—	—
78	19,462	79.56	3,378.59	41	13,955	—	—	—	—	—	—	—	—	—
79	114,999	70.12	2,127.41	21	8,811	2,959.92	29	12,824	—	—	—	—	—	—

Table A.20 (cont. - Serial comparison Cicirello)

		Tanaka	SPHP+PH			SPHP			MF-SPHP-PH			MF-SPHP		
ins	val	time	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols	time	#nodes	#cols
81	383,485	30.13	737.00	1	4,974	—	—	—	743.40	1	4,974	2,662.26	13	16,216
86	361,417	68.14	1,151.50	19	8,984	1,364.22	23	10,619	—	—	—	—	—	—
96	455,448	79.50	1,221.29	13	5,434	1,353.00	15	5,941	—	—	—	—	—	—
97	407,590	92.92	2,525.89	29	11,851	—	—	—	—	—	—	—	—	—
98	520,582	90.23	1,113.84	19	5,070	1,110.57	19	5,070	—	—	—	—	—	—
99	363,518	106.21	922.87	9	4,081	2,155.09	23	10,613	—	—	—	—	—	—
100	431,736	80.98	2,302.93	19	9,966	—	—	—	—	—	—	—	—	—
101	352,990	58.17	2,122.52	25	14,025	—	—	—	—	—	—	—	—	—
102	492,572	55.05	2,604.45	29	17,775	3,126.29	37	21,535	—	—	—	—	—	—
103	378,602	46.02	2,225.61	27	15,748	3,198.39	43	24,234	—	—	—	—	—	—
105	450,806	41.61	3,456.39	43	24,070	—	—	—	—	—	—	—	—	—
106	454,379	56.18	3,500.56	49	22,629	—	—	—	—	—	—	—	—	—
107	352,766	41.31	1,422.14	11	9,953	2,980.28	29	21,909	—	—	—	—	—	—
108	460,793	39.30	2,143.26	21	14,973	2,162.82	21	14,973	—	—	—	—	—	—
112	367,110	114.06	3,219.53	57	14,379	3,558.68	63	16,420	—	—	—	—	—	—
114	463,474	107.27	3,165.91	37	13,676	—	—	—	—	—	—	—	—	—
120	396,183	84.73	1,248.65	13	5,495	2,692.26	23	12,362	—	—	—	—	—	—

Appendix B. Detailed Speedup Results of Computational Experiments

This section of the appendix presents detailed results on the speedup of the parallelization approaches (SPBP-PH and HPBP-PH). The following tables B.21–B.26 show summary speedup statistics for both versions of the algorithm and for the three benchmark sets. The first column contains the algorithm acronym, the second column contains the abbreviated name of the statistical metric, followed by the metric values over the number of threads used in the algorithm. The indicators include medians (Med), averages (Avg), standard deviations (SD), and the coefficients of variation (CV; which is the ratio of the SD and Avg). They were taken from instances in the respective benchmark set that took at least 60 seconds to solve, to account for cases where no parallelization (or even column generation) was used, e.g. when the algorithm terminated with the initial schedule found by the heuristic.

This is followed by visualizations of the same data as box plots in figures B.2–B.4. Finally, plots of the parallel efficiency for the algorithms are shown in figures B.5, and table B.27 shows the comparison of waiting conditions during the execution of the two algorithms (see Section 5).

Table B.21: Average overall parallel speedup (Rubin)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.58	2.53	4.40	5.27	6.16	7.04
SPBP	Avg	1.00	1.58	2.48	4.18	5.26	6.14	6.85
-PH	(SD)	(0.00)	(0.04)	(0.30)	(0.44)	(0.48)	(0.74)	(1.07)
	(CV)	(0.00)	(0.03)	(0.12)	(0.11)	(0.09)	(0.12)	(0.16)
	Med	1.00	1.73	2.89	4.47	5.46	6.30	7.10
HPBP	Avg	1.00	1.73	2.78	4.42	5.47	5.89	7.10
-PH	(SD)	(0.00)	(0.07)	(0.37)	(0.59)	(0.56)	(1.11)	(1.34)
	(CV)	(0.00)	(0.04)	(0.13)	(0.13)	(0.10)	(0.19)	(0.19)

Table B.22: Average overall parallel speedup (Gagné)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.64	2.80	4.41	5.86	7.74	8.93
SPBP	Avg	1.00	1.64	2.82	4.62	6.10	7.89	8.87
-PH	(SD)	(0.00)	(0.12)	(0.30)	(0.73)	(1.14)	(1.79)	(2.56)
	(CV)	(0.00)	(0.07)	(0.11)	(0.16)	(0.19)	(0.23)	(0.29)
	Med	1.00	1.77	2.93	4.75	6.13	7.54	9.16
HPBP	Avg	1.00	1.78	2.97	4.83	6.46	8.63	9.88
-PH	(SD)	(0.00)	(0.15)	(0.58)	(0.99)	(1.35)	(3.81)	(3.46)
	(CV)	(0.00)	(0.09)	(0.19)	(0.20)	(0.21)	(0.44)	(0.35)

Table B.23: Average overall parallel speedup (Cicirello)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.70	2.86	4.37	5.90	7.19	7.81
SPBP	Avg	1.00	1.68	2.86	4.36	5.73	7.04	9.78
-PH	(SD)	(0.00)	(0.12)	(0.37)	(0.86)	(1.28)	(1.85)	(21.11)
	(CV)	(0.00)	(0.07)	(0.13)	(0.20)	(0.22)	(0.26)	(2.16)
	Med	1.00	1.74	2.89	4.37	5.89	7.51	8.29
HPBP	Avg	1.00	1.71	2.86	4.46	11.44	9.17	12.57
-PH	(SD)	(0.00)	(0.19)	(0.58)	(1.92)	(41.60)	(14.69)	(41.33)
	(CV)	(0.00)	(0.11)	(0.20)	(0.43)	(3.64)	(1.60)	(3.29)

Table B.24: Average parallel DP speedup (Rubin)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.67	2.95	6.35	10.38	15.72	22.89
SPBP	Avg	1.00	1.67	2.96	6.07	10.15	15.33	22.54
-PH	(SD)	(0.00)	(0.04)	(0.34)	(0.60)	(1.01)	(2.45)	(5.35)
	(CV)	(0.00)	(0.02)	(0.12)	(0.10)	(0.10)	(0.16)	(0.24)
	Med	1.00	1.86	3.51	6.72	11.22	16.92	25.05
HPBP	Avg	1.00	1.85	3.39	6.57	10.96	15.45	25.93
-PH	(SD)	(0.00)	(0.06)	(0.41)	(0.62)	(0.97)	(4.01)	(5.55)
	(CV)	(0.00)	(0.03)	(0.12)	(0.09)	(0.09)	(0.26)	(0.21)

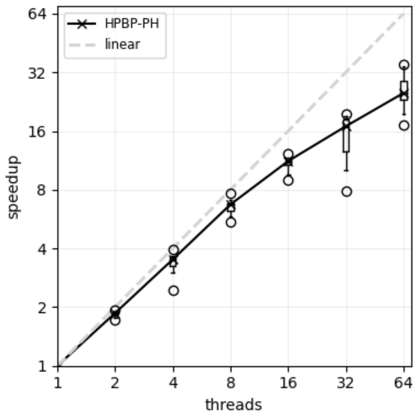
Table B.25: Average parallel DP speedup (Gagné)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.74	3.35	6.41	10.60	18.74	32.63
SPBP	Avg	1.00	1.73	3.30	6.36	10.78	18.51	29.39
-PH	(SD)	(0.00)	(0.10)	(0.27)	(0.49)	(0.86)	(2.31)	(6.36)
	(CV)	(0.00)	(0.06)	(0.08)	(0.08)	(0.08)	(0.12)	(0.22)
	Med	1.00	1.91	3.61	7.05	12.21	21.35	35.44
HPBP	Avg	1.00	1.90	3.54	6.89	12.11	21.52	34.42
-PH	(SD)	(0.00)	(0.10)	(0.61)	(0.79)	(1.12)	(3.95)	(6.78)
	(CV)	(0.00)	(0.05)	(0.17)	(0.12)	(0.09)	(0.18)	(0.20)

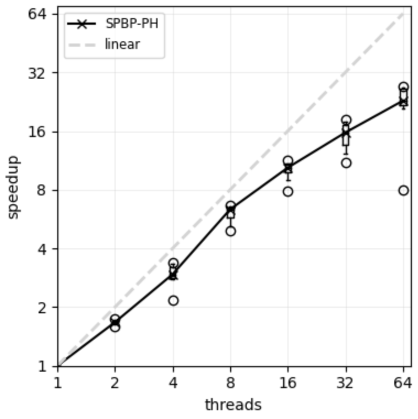
Table B.26: Average parallel DP speedup (Cicirello)

#threads		1	2	4	8	16	32	64
	Med	1.00	1.82	3.54	6.70	11.76	19.84	28.46
SPBP	Avg	1.00	1.83	3.56	6.69	11.75	19.93	35.52
-PH	(SD)	(0.00)	(0.11)	(0.29)	(0.68)	(1.14)	(3.66)	(76.87)
	(CV)	(0.00)	(0.06)	(0.08)	(0.10)	(0.10)	(0.18)	(2.16)
	Med	1.00	1.90	3.61	6.79	12.28	20.44	30.40
HPBP	Avg	1.00	1.87	3.62	7.10	27.95	27.25	50.93
-PH	(SD)	(0.00)	(0.19)	(0.67)	(3.14)	(116.36)	(45.94)	(190.33)
	(CV)	(0.00)	(0.10)	(0.19)	(0.44)	(4.16)	(1.69)	(3.74)

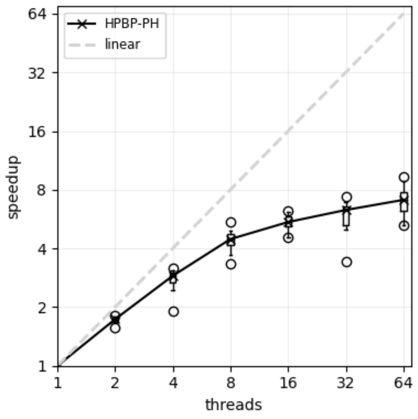
Figure B.2: Box plots for parallel Speedups (Rubin)



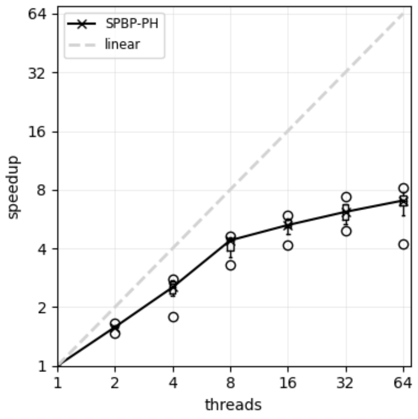
(a) Parallel Speedup DP (Rubin, HPBP-PH)



(b) Parallel Speedup DP (Rubin, SPBP-PH)

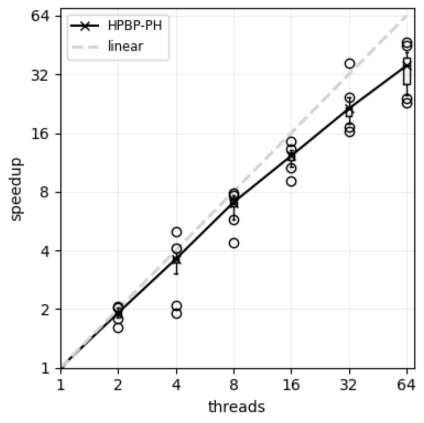


(c) Overall parallel Speedup (Rubin, HPBP-PH)

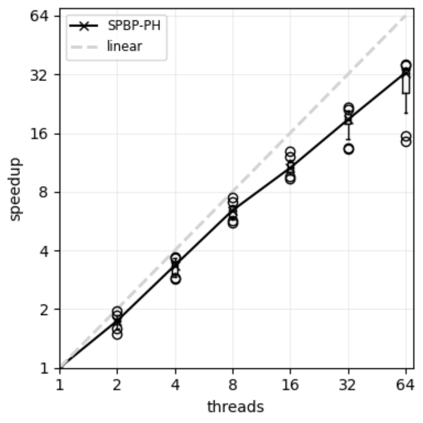


(d) Overall parallel Speedup (Rubin, SPBP-PH)

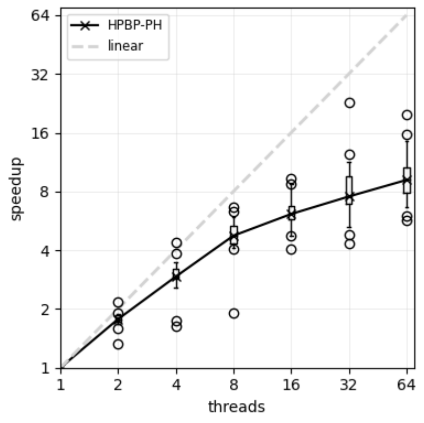
Figure B.3: Box plots for parallel Speedups (Gagné)



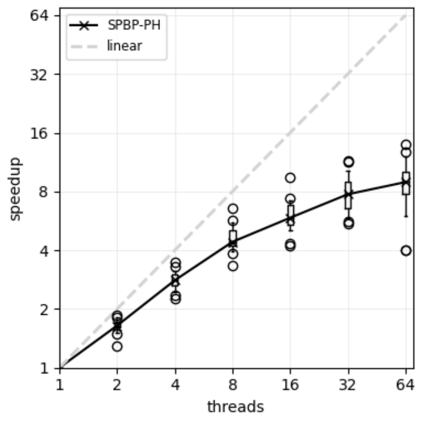
(a) Parallel Speedup DP (Gagné, HPBP-PH)



(b) Parallel Speedup DP (Gagné, SPBP-PH)

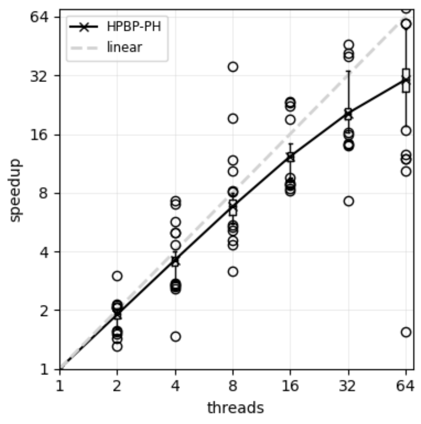


(c) Overall parallel Speedup (Gagné, HPBP-PH)

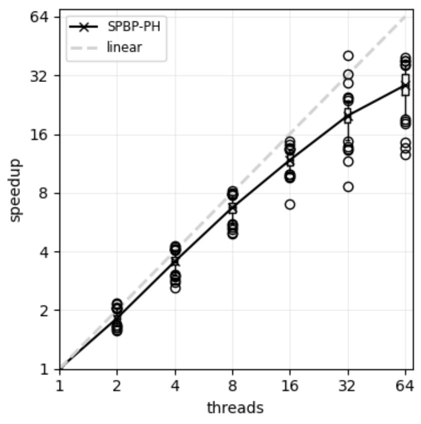


(d) Overall parallel Speedup (Gagné, SPBP-PH)

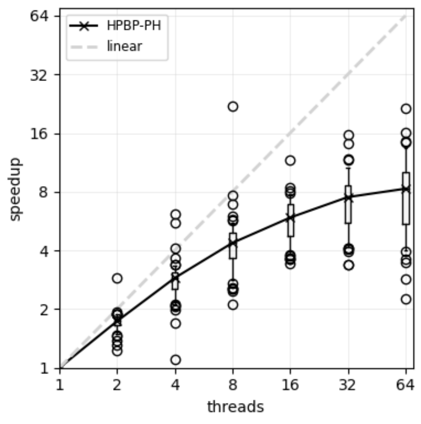
Figure B.4: Box plots for parallel Speedups (Cicirello)



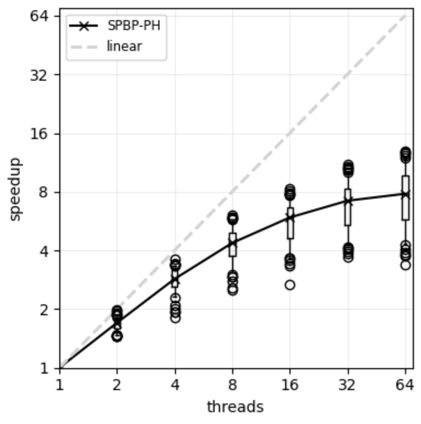
(a) Parallel Speedup DP (Cicirello, HPBP-PH)



(b) Parallel Speedup DP (Cicirello, SPBP-PH)

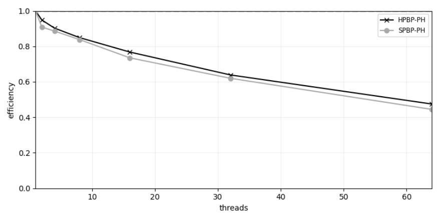


(c) Overall parallel Speedup (Cicirello, HPBP-PH)

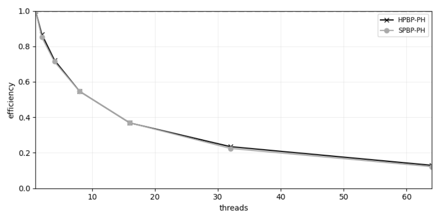


(d) Overall parallel Speedup (Cicirello, SPBP-PH)

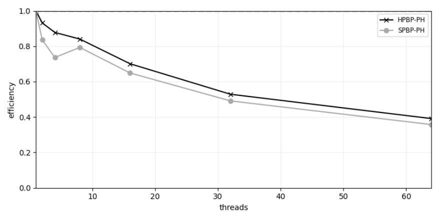
Figure B.5: Average parallel efficiencies



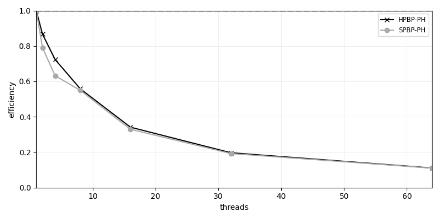
(a) Parallel efficiency DP (Cicirello)



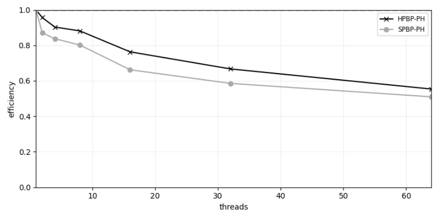
(b) Overall parallel efficiency (Cicirello)



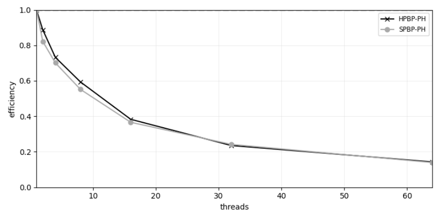
(c) Parallel efficiency DP (Rubin)



(d) Overall parallel efficiency (Rubin)



(e) Parallel efficiency DP (Gagné)



(f) Overall parallel efficiency (Gagné)

Table B.27: Ratio of average number of times entering the waiting condition in SPBP-PH and HPBP-PH for 2 to 64 threads (no waiting in single-core execution)

$\frac{\varnothing\ \#\text{wait SPBP-PH}}{\varnothing\ \#\text{wait HPBP-PH}}$	2	4	8	16	32	64	$\varnothing$
Rubin	126	317	86	1	5	0	76
Gagné	541	219	92	137	287	2	182
Cicirello	47	24	588	5	28	54	107

Appendix C. Recomputed Results of Tanaka and Araki's Experiments

We re-enacted all the experiments in Tanaka and Araki (2013) on the same hardware as the experiments in section 6. In their study, the authors used different settings for the algorithm depending on the instances to which it was applied to. The first setting was applied to instances Nos. 41–120 of the Cicirello set and the second one to the Rubin, Gagné and the rest of the Cicirello set, using different amounts of memory to store their network structure for what the authors call the "second stage" (denoted by "512MB", "2GB" and "20GB" in the tables below). For the specifics, see Tanaka and Araki (2013).

To replicate the computational environment as closely as possible, we conducted the experiments with the implementation unchanged, applied preprocessing to instances Nos. 1–40 of the Cicirello set (removing zero-weight jobs under certain conditions), and used the same settings reported by the authors. The two main differences from the originally reported data are hardware and the fact that the cluster didn't allow us to run programs for more than 21 days. Since the instances in question (except for Cicirello No. 18 with 14 days) took the authors more than 30 days to solve (using "20GB" memory), we ran them for 14 days. The timelimit affects the results for the following instances (with the running times from Tanaka and Araki (2013) in parentheses):

- 751 (34 days), 851 (> 30 days; unsolved) and 855 (> 30 days; unsolved) from the Gagn'e set; and
- Nos. 18 (2 weeks) and 24 (30 days) from the Cicrello set.

Note, that we set a hard time limit of 14 days, so instance No. 18 may have taken only another hour to solve and prove optimality.

The detailed running times from the re-run experiments are given in Tables C.28–C.32. The first three columns in Tables C.28 and C.29 give the name of the instance from the Rubin and Gagné set, the number of jobs in the instance, and the optimal or best known objective value for the instance. The following three columns show the running times for the instance from the re-enacted experiments on the cluster, using different amounts of memory for the “second stage” of the algorithm as mentioned above. In case the re-run experiment did not terminate within the timelimit, the second column also shows the original objective value (from the original experiments in their study) in parentheses where applicable. Also, provably optimal objective values are highlighted as bold numbers.

Table C.30 gives the same information for Cicirello instances Nos. 1–40 as the previous two tables, only omitting the column with the number ob jobs, since all instances from this set contain the same number of jobs (60).

Finally, Tables C.31 and C.32 address instances Nos. 41–80 and Nos. 81–120 of the Cicirello set, respectively. Both tables contain columns for the name of the instance, the optimal value, and a single column for the running time, since no increased amount of memory was required for these instances.

Table C.28: Runningtimes Tanaka and Araki on Rubin instances

Name	jobs	Opt	512MB	2GB	20GB
401	15	**90**	0.34	0.34	0.31
402	15	**0**	0.01	0.00	0.01
403	15	**3,418**	0.56	0.54	0.53
404	15	**1,067**	0.46	0.48	0.44
405	15	**0**	0.00	0.00	0.00
406	15	**0**	0.00	0.01	0.01
407	15	**1,861**	0.48	0.48	0.48
408	15	**5,660**	0.87	0.85	0.85
501	25	**261**	2.18	2.14	2.09
502	25	**0**	0.01	0.01	0.01
503	25	**3,497**	3.10	3.08	3.04
504	25	**0**	0.03	0.03	0.03
505	25	**0**	0.02	0.02	0.02

Table C.28: cont. - Runningtimes Tanaka and Araki on Rubin instances

Name	jobs	Opt	512MB	2GB	20GB
506	25	**0**	0.02	0.02	0.02
507	25	**7,225**	4.27	4.28	4.22
508	25	**1,915**	3.86	3.84	3.88
601	35	**12**	7.48	7.36	7.31
602	35	**0**	0.05	0.05	0.05
603	35	**17,587**	18.37	20.33	19.73
604	35	**19,092**	25.73	25.74	27.32
605	35	**228**	13.84	13.57	14.90
606	35	**0**	0.05	0.05	0.05
607	35	**12,969**	18.32	17.95	17.60
608	35	**4,732**	33.33	33.09	32.92
701	45	**97**	49.55	60.15	55.34
702	45	**0**	0.10	0.09	0.09
703	45	**26,506**	53.58	53.71	52.36
704	45	**15,206**	72.01	72.31	71.40
705	45	**200**	866.47	599.48	862.37
706	45	**0**	0.11	0.10	0.11
707	45	**23,789**	53.35	53.63	52.32
708	45	**22,807**	100.73	100.58	99.77

Table C.29: Runningtimes Tanaka and Araki on Gagné instances

Name	jobs	Opt	512MB	2GB	20GB
551	55	**183**	116.23	150.63	282.44
552	55	**0**	0.18	0.17	0.17

Table C.29: cont. - Runningtimes Tanaka and Araki on Gagné instances

Name	jobs	Opt	512MB	2GB	20GB
553	55	**40,498**	116.16	114.99	112.04
554	55	**14,653**	277.87	281.62	278.99
555	55	**0**	0.30	0.27	0.28
556	55	**0**	0.19	0.18	0.18
557	55	**35,813**	146.85	146.07	140.91
558	55	**19,871**	229.22	230.05	228.58
651	65	**247**	26,575.61	4,611.69	2,298.27
652	65	**0**	0.25	0.25	0.26
653	65	**57,500**	343.74	304.17	262.80
654	65	**34,301**	390.76	381.56	389.30
655	65	**0**	355.72	430.00	1,816.32
656	65	**0**	0.30	0.29	0.30
657	65	**54,895**	263.95	261.66	253.71
658	65	**27,114**	471.35	469.43	466.63
751	75	225 (**225**)	—	—	—
752	75	**0**	0.45	0.44	0.45
753	75	**77,544**	839.59	598.87	565.95
754	75	**35,200**	907.44	888.11	887.85
755	75	**0**	0.42	0.41	0.41
756	75	**0**	0.54	0.54	0.52
757	75	**59,635**	1,197.81	844.79	741.33
758	75	**38,339**	998.00	991.69	987.59
851	85	363 (360)	—	—	—
852	85	**0**	0.62	0.63	0.61
853	85	**97,497**	2,519.42	2,219.22	6,188.71
854	85	**79,042**	1,451.49	1,222.28	1,177.04

Table C.29: cont. - Runningtimes Tanaka and Araki on Gagné instances

Name	jobs	Opt	512MB	2GB	20GB
855	85	260 (260)	—	—	—
856	85	**0**	0.63	0.64	0.63
857	85	**87,011**	6,130.77	4,594.41	4,897.54
858	85	**74,739**	1,867.57	1,850.23	1,817.68

Table C.30: Runningtimes Tanaka and Araki on Cicirello instances 1–40

Name	Opt	512MB	2GB	20GB
1	**453**	120.06	101.78	93.58
2	**4,794**	4,841.34	1,499.98	1,474.67
3	**1,390**	2,173.60	1,000.58	589.07
4	**5,866**	206.60	169.31	129.80
5	**4,054**	3,563.57	2,350.37	2,768.09
6	**6,592**	166.86	243.35	171.90
7	**3,267**	8,640.60	3,522.47	6,680.84
8	**100**	122.22	109.55	105.54
9	**5,660**	139.24	130.39	124.33
10	**1,740**	16,455.70	9,783.92	8,883.74
11	**2,785**	—	120,893.73	42,921.26
12	**0**	0.42	0.43	0.41
13	**3,904**	15,674.95	8,722.61	9,907.07
14	**2,075**	20,705.47	8,764.03	3,555.71
15	**724**	2,802.18	841.15	496.78
16	**3,285**	840.15	802.80	504.49

Table C.30: cont. - Runningtimes Tanaka and Araki on Cicirello instances 1–40

Name	Opt	512MB	2GB	20GB
17	**0**	1,189.80	2,767.39	5,696.84
18	773 (**767**)	—	—	—
19	**0**	3.41	3.47	3.42
20	**1,757**	762.29	334.69	316.22
21	**0**	0.22	0.20	0.21
22	**0**	0.23	0.25	0.23
23	**0**	0.17	0.18	0.17
24	761 (**761**)	—	—	—
25	**0**	0.32	0.33	0.31
26	**0**	0.23	0.25	0.24
27	**0**	0.35	0.36	0.34
28	**0**	0.40	0.40	0.46
29	**0**	0.24	0.23	0.23
30	**0**	8.00	8.14	7.85
31	**0**	0.47	0.48	0.45
32	**0**	0.51	0.51	0.48
33	**0**	0.50	0.51	0.48
34	**0**	0.40	0.43	0.44
35	**0**	0.47	0.47	0.50
36	**0**	0.44	0.46	0.48
37	**0**	2,890.32	595.89	2,542.07
38	**0**	0.38	0.38	0.40
39	**0**	0.48	0.48	0.53
40	**0**	0.41	0.40	0.43

Table C.31: Runningtimes Tanaka and Araki on Cicirello instances 41–80

Name	Opt	Time	Name	Opt	Time
41	**69,102**	32.06	61	**75,916**	41.42
42	**57,487**	45.69	62	**44,769**	37.31
43	**145,310**	80.55	63	**75,317**	33.11
44	**35,166**	58.20	64	**92,572**	39.94
45	**58,935**	80.98	65	**126,696**	41.08
46	**34,764**	54.08	66	**59,685**	17.81
47	**72,853**	61.35	67	**29,390**	26.93
48	**64,612**	105.70	68	**22,120**	25.99
49	**77,449**	59.38	69	**71,118**	45.03
50	**31,092**	59.55	70	**75,102**	36.69
51	**49,208**	91.84	71	**145,007**	96.11
52	**93,045**	117.65	72	**43,286**	68.28
53	**84,841**	116.94	73	**28,785**	91.49
54	**118,809**	104.63	74	**29,777**	73.14
55	**64,315**	108.07	75	**21,602**	77.57
56	**74,889**	122.09	76	**53,555**	75.75
57	**63,514**	97.11	77	**31,817**	84.48
58	**45,322**	135.31	78	**19,462**	79.56
59	**50,999**	92.49	79	**114,999**	70.12
60	**60,765**	137.23	80	**18,157**	76.73

Table C.32: Runningtimes Tanaka and Araki on Cicirello instances 81–120

Name	Opt	Time	Name	Opt	Time
81	**383,485**	30.13	101	**352,990**	58.17
82	**409,479**	66.12	102	**492,572**	55.05
83	**458,752**	42.51	103	**378,602**	46.02
84	**329,670**	46.59	104	**357,963**	56.07
85	**554,766**	74.38	105	**450,806**	41.61
86	**361,417**	68.14	106	**454,379**	56.18
87	**398,551**	47.96	107	**352,766**	41.31
88	**433,186**	60.54	108	**460,793**	39.30
89	**410,092**	46.85	109	**413,004**	54.97
90	**401,653**	56.97	110	**418,769**	60.62
91	**339,933**	68.98	111	**342,752**	101.73
92	**361,152**	103.36	112	**367,110**	114.06
93	**403,423**	125.52	113	**259,649**	107.98
94	**332,941**	98.30	114	**463,474**	107.27
95	**516,926**	101.20	115	**456,890**	115.58
96	**455,448**	79.50	116	**530,601**	107.89
97	**407,590**	92.92	117	**502,840**	117.72
98	**520,582**	90.23	118	**349,749**	62.23
99	**363,518**	106.21	119	**573,046**	121.78
100	**431,736**	80.98	120	**396,183**	84.73