WHICH CATEGORIES DO PROSPECTIVE TEACHERS APPLY WHEN SELECTING TASKS FOR THE SCALAR PRODUCT?

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Selecting tasks is a typical job in everyday teaching that requires teachers to have content-specific expertise, including categories to think about and perceive conceptual understanding. This contribution investigates the activation of prospective teachers' categories when solving tasks on the scalar product and evaluating their relevance for teaching in upper secondary school. To further investigate, whether input on content-related aspects could be helpful to promote a professional justification for the selection of tasks, two groups with and without input are to be differentiated. Different ways of dealing with underlying concept elements and evaluating the relevance of tasks are discussed by examples.

INTRODUCTION AND THEORETICAL BACKGROUND

To select tasks for lessons professionally, teachers acquire content-specific expertise, which can be specified as "the personal capacity to cope with situational demands called jobs" (Prediger, 2019, p. 371). Content-specific categories are part of this expertise and are defined as "conceptual […] knowledge elements that filter and focus the categorical perception and the thinking of the teachers" (Prediger, 2019, p. 370). In the presented study, the job of selecting tasks for initiating development of conceptual understanding of the scalar product is to be examined. The scalar product is a content of vectoral analytical geometry in German upper secondary school.

Aspects teachers focus on while considering a task can regard different elements of instruction like the content or the goal of the task, or more specific information about the lesson context (González et al., 2020). As various studies such as PISA and TIMSS show that conceptual understanding is often neglected in Germany, the principle of conceptual focus, which underlines the understanding of both concepts and procedures (Prediger et al., 2022), is placed at the centre here. In the following, prospective teachers' categories relating to a conceptual focus are specified to describe how they select tasks for teaching the scalar product. An example of using categories to investigate in content-specific jobs can be found in Prediger and Zindel (2017).

Categories referring to conceptual understanding of the scalar product

From the perspective of cognitive theory, conceptual understanding can be defined as the construction of flexible and transparent structures, which is the theoretical assumption for a model of conceptual understanding according to Drollinger-Vetter

2025. In C. Cornejo et al. (Eds.). *Proceedings of the 48th Conference of the International Group for the Psychology of Mathematics Education*, Vol. X, pp. XX-YY. PME.

(2011). In this model, *underlying concept elements* form the basis for conceptual understanding and must be explicitly addressed in a comprehensible lesson to enable students to deal with different types of representation. Then, understanding a concept manifests itself in the *unfolding and condensing* of concept elements, representations and the concept itself, which is also linked to other concepts. Fig. 1 shows the model, which needs to be specified in terms of the scalar product (Herrmann, 2025). Elements of the three different layers can then be identified as content-specific categories, marked with <...>.

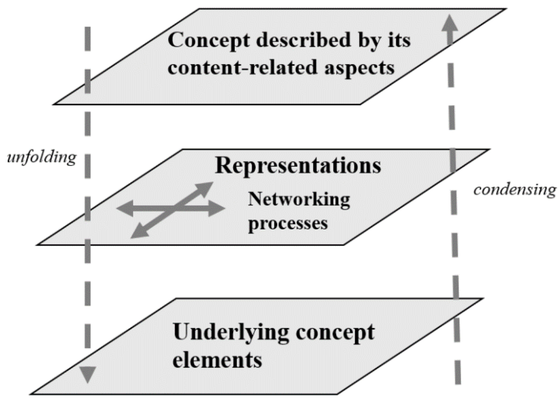

Fig. 1. Model of conceptual understanding based on Drollinger-Vetter (2011, p.190)

An initial point of reference for the didactic reconstruction of mathematical content is provided by **content-related aspects** that can be found at the top layer in Fig. 1. In the case of the scalar product, the following four aspects (Hoffmann et al., in prep.) are found to be relevant to teaching: a) The aspect of <*product operation*> describes the scalar product as an operation between two vectors. b) The aspect of <*monotonicity in angle and length changes*> relates to the dependence of the scalar product on elementary geometric quantities. c) The aspect of <*static angle*> is based on qualitative and quantitative statements about the angles involved. Finally, d) the aspect of <*equal scalar products*> makes a geometric statement about vectors with equal scalar products.

To make the scalar product understandable by having those aspects in mind, content-specific representations are required, which can be found on the middle layer in Fig. 1. In the area of vectoral analytical geometry, geometrical and algebraical representations are usually distinguished, whereby both can be used with or without vectors. This results in four **forms of representations**: <*elementary geometry*>, <*arrow geometry*>, <*elementary algebra*> and <*vector algebra*> (Herrmann, et al., in prep.). These representations can be networked with each other through various processes. For example, to convert a geometric representation into an algebraic representation, an <*algebraisation*> is required, which is a result of applying "the power of algebra to geometry" (Dorier, 2000, p.13). Other kinds of those **networking processes** in the field of analytical geometry are <*geometrisation*>, as well as <*vectorisation*> and <*elementary interpretation*> (Herrmann, et al., in prep.).

Unfolding these representations, content-related **underlying concept elements** can be identified and categorised in the bottom layer in Fig. 1. Examples for these elements, which relate to different kinds of representations, are *angles, lengths, arrows, projections, cosinus* and *n-tuples.* While *angles* and *lengths* are an example of

<elementary geometric elements>, arrows can be described as *<arrow geometrical elements>*, trigonometric concepts such as *cosinus* as *<elementary algebraic elements>* and *n-tuples* as *<vector algebraic elements>*.

Insightful handling of categories

To successfully condense and unfold the category elements on the three different layers, it is necessary to deal with them in such a way that they have a meaning for the concept of the scalar product addressed in the tasks. According to Lotz (2021), there is a difference between a naïve and an insightful approach to these elements. For example, an angle between two vectors could be determined naively by looking up a formula, plugging in the known variables and calculating the magnitude of the angle. An *insightful handling* of underlying concept elements, representations or content-related aspects is characterised by the fact that a person sees it, what they are supposed to see in it in relation to the concept of the scalar product (Lotz, 2022, p. 199).

RESEARCH GAP AND RESEARCH QUESTIONS

Focussing on the selection of tasks as a typical job for teaching mathematics, it is known that task-centric approaches improve teachers' ability to select cognitively challenging tasks (Boston & Smith, 2011). It is not yet clear whether an input on content-related aspects also makes a difference to the categories teachers activate. Regarding teachers' expertise, Prediger and Zindel (2017) found out that an intervention on certain categories has an influence on how teachers deal with content-specific jobs, but no such study had been carried out for the selection of tasks on the scalar product.

This contribution therefore examines the extent to which prospective teachers activate content-specific categories for conceptual understanding when selecting tasks on the scalar product. Therefore, a deeper look is taken at how prospective teachers solve tasks themselves and discuss their relevance for students afterwards.

(RQ1) Which content-specific categories can be reconstructed when prospective teachers solve tasks and assess their relevance for teaching the scalar product depending on whether they have received input on certain categories.

(RQ2) How do prospective teachers justify the selection of tasks for teaching the scalar product depending on whether they have received input on certain categories?

Based on Prediger and Zindel (2017, p. 235) it can be conjectured that the categories mentioned in the input are activated more often than it would be the case without input.

METHODS

Data collection: The data on which this study is based was collected in two classes for prospective secondary teachers at Paderborn University. At the beginning of the course, participants of one seminar received a 15-minute input on the scalar product (*intervention group*), while those in the other seminar did not (*control group*). The input is located in the top and middle layer of the model of conceptual understanding, as the content-related aspects (*<product operation>, <monotonicity in angle and length changes>, <static angle>* and *<equal scalar products>)* are given. To visualise and describe the aspects, different representations are used, e.g. a geometric representation of arrows with equal scalar products (Fig 2.). Underlying concept elements and general didactical principles haven't been addressed explicitly. Then, both classes were given 30 minutes time to solve the three different tasks and evaluate their relevance for teaching the scalar product: The first task deals with the aspect of orthogonality. The aspect of equal scalar products is addressed in task 2 by drawing different arrows with the same scalar product in relation to $\vec{v}$. In the third task, a three-dimensional angle must be computed using the scalar product as a measure of geometric quantities. All three task can be found here: go.upb.de/tasks

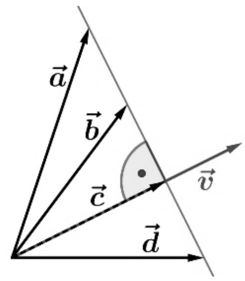

Fig. 2. Geometric representation of equal scalar products

Sample: Data was collected from a total of n=26 participants, which had been divided into six break-out groups (each 3 groups with and without intervention), consisting of 3-6 participants. A total of 180 minutes of video material was produced and completely transcribed. The written results are also available. All of the participants study in a master's programme to become mathematics teachers for secondary school. Most of them are in their third or fourth master's semester and have already completed most of the courses as well as a six-month school practical training. Overall, it is assumed that the participants have the same formal prerequisites. The courses have been put together at random so that natural heterogeneity is created.

Data analysis: As part of an evaluative qualitative content analysis (Mayring, 2014), the content-specific categories of the model of conceptual understanding were identified and coded in the transcripts in three steps. In a first step, all statements in the transcripts were categorised according to whether they relate to the solution of the task or to the evaluation of relevance. Then, the above-mentioned categories were coded in relation to both the task solution and the evaluation. A compilation and comparison of all coded elements shows that a further differentiation must be made in the coding system regarding *how* the prospective teachers deal with the various coded elements. Therefore, all coded elements were analysed in a third step to determine whether they had been used insightfully. The result was a document in which only the category elements that could be identified as insightful were coded. All elements that were used naively or could not be clearly assigned were omitted from the second coding cycle. The inter-coder reliability of the coded elements has been ensured by an open discussion. About 5% of the coded material of the first coder have been discussed with different second coders (Mayring, 2014, p.114). By comparing the analysed data, both quantitative and qualitative phenomena can be outlined descriptively.

RESULTS

While participants of the intervention group spent around 59% of their 30 minutes time on solving the tasks, the average in the control group was 72%. The greatest difference in time was found when working on task 2, where arrows with the same scalar product in relation to $\vec{v}$ had to be found. The following analyses provide a more detailed insight into the prospective teachers' solution and assessment processes for task 2 and contrast participants in the intervention and control group.

Insightfully activated categories when solving and assessing tasks

The following transcript excerpt comes from participants of the control group and shows a discussion that arises during their process of solving task 2. They tried to find equal scalar products by drawing perpendicular arrows to $\vec{v}$ before but recognised that they can’t find three arrows with different directions that fulfil these requirements.

1 Anne: Isn’t the scalar product the angle between those [arrows]. Or has anything to do with // angles?

2 Mick: It’s an area.

3 Theo: If it’s orthogonal, it is 0.

4 Mick: Yes, but let’s think about it – three different directions

5 Anne: Then we can make it with // three different angles.

<*Elementary geometric elements*> like *angle* and *area* and <*arrow geometric elements*> like *direction* can be identified. Besides, Anne deals with the content-related aspect of <*orthogonality*> to confirm Theo's statement that the scalar product has something to do with angles, at least in special cases. The idea of linking directions of arrows with angles can be described as <*vectorisation*>. What makes this transcript section remarkable is that none of the coded categories is used *insightfully* to solve the given task. As far as the model of conceptual understanding is concerned, the group isn’t able to condense underlying conceptual elements into meaningful representations.

Since the intervention group previously got to know a geometrical representation of equal scalar products (fig. 2), all break-out groups were able to reconstruct the sketch in relation to the arrow $\vec{v}$ and evaluate it as the correct answer. The following excerpt shows a discussion of such a break-out group on how students might deal with the task.

1 Lea: I think a student can only solve the task, if this sketch is known. And then it doesn’t make sense, does it?

2 Tom: Yes, but if you know that you are projecting orthogonally from one vector onto the other… and then take the length of the part, where the orthogonal projection occurs. So, you could probably work it out, but you need to understand how the scalar product works with the orthogonal projection and not just how to calculate it.

Again, various <*geometric elements*> can be identified, such as *projection, orthogonal* and *length*, but unlike the other group, they are used *insightfully*, as they are meaningfully linked to the given representation. Tom obtains these concept elements from *unfolding* the given representation, which is remarkable, because the concept elements were not explicitly given in the input.

The two transcripts are shown as examples for the failure of *condensing* on the one hand and successful *unfolding* on the other, which turned out to be a difference between the control and intervention group working on task 2. For the control group, the aspect of <*orthogonality*> is the first point of reference for the scalar product, which works for task 1, but does not apply to task 2. Although many more <*networking processes*> have been coded in transcripts of the control group, the fewest of them are used *insightfully* to solve the problem.

Adding up the number of coded underlying concept elements both in relation to task solving and evaluating the relevance of task 2, the control group used 28% *insightfully*, while the intervention group achieved a percentage of 45%. This can be partly explained by the given input, but it is worth mentioning that the underlying concept elements were not explicitly addressed in the input. Table 1 shows the absolute numbers of all coded elements and the number of those used insightfully.

Underlying concept elements	Intervention group		Control group	
	all	insightful	all	insightful
<*Elementary geometric elements*>	19	5	22	4
<*Elementary algebraic elements*>	2	2	30	14
<*Arrow geometric elements*>	28	10	64	13
<*Vector algebraic elements*>	1	1	13	5

Table 1: Coded underlying concept elements for task 2

In addition, the data in table 1 shows that the control group used lots of <*elementary algebraic elements*> and <*vector algebraic elements*> when working on task 2. Their strategy to solve the task, coded as an <*algebraisation*>, is mostly to read off the coordinates of the given arrow $\vec{v}$ and try to solve the task algebraically by setting up a system of equations to find vectors with the same scalar product. Although this type of strategy does not require geometric knowledge of equal scalar products, only one out of three break-out control groups succeeded in finding different vectors with the same scalar product. The group that found different vectors algebraically was not even able to specify an <*arrow geometric representation*> of equal scalar products, which is why not a single one of those was coded in the control groups' transcripts. The intervention group, on the other hand, did not need this type of <*algebraisation*>, as they had become familiar with an <*arrow geometric representation*> of equal scalar products in the input.

Prospective teachers‘ justification of selected tasks for teaching

In the above transcript excerpt, Tom (intervention group) points out that he associates conceptual understanding of the scalar product with projections and distinguishes it from mere computation. For solving task 2, he, therefore, emphasises the need for conceptual understanding, which he underpins with <*Elementary geometric elements*> *and* <*arrow geometric elements*>. Anne, Theo, Mia and Mick (control group) noticed the conceptual focus of task 2 as well, but used it on a *surface-level* (Lotz, 2022, p.120):

1	Anne:	How relevant do you find the tasks? […]
2	Theo:	12 from 10 [points]. All of them are relevant for the final examination.
3	Mia:	For conceptual understanding, I guess?
4	Mick:	Task 2 is best for that, isn't it? […] But you don't understand that the scalar product is an area. That doesn't even happen in school lessons.

It should be emphasised here that, although Mia and Mick are aware of the conceptual focus of task 2, they are not able to work it out in a content-specific way. Mick actually brings in some sort of conceptual understanding that he assumes to be relevant to the scalar product, but he does not notice a connection to the task, like Tom from the intervention group does in the transcript section above. That is why Mia's and Mick's contribution is not rated *insightfully*. Such phenomena can also be observed in other parts of the data, particularly in relation to the other two tasks.

CONCLUSION AND OUTLOOK

To summarise and answer RQ1, the aspect of <*equal scalar products*> with the associated <*arrow geometric representation*> and underlying <*elementary geometric elements*> and <*arrow geometric elements*> could be descriptively reconstructed in solving and evaluation processes of prospective teachers from the intervention group. It was found that prospective teachers of the control group use fewer categories related to the conceptual understanding model in an *insightful* way and that, in addition, there is an emphasis on <*elementary algebraic elements*> and <*vector algebraic elements*> instead. For both solving and evaluating a task for teaching the scalar product, it is noticeable that participants of the intervention group seem to be more successful in *unfolding and condensing* in the sense of the model of conceptual understanding.

About the justification of selected tasks for teaching, the analysis of various transcript excerpts shows that prospective teachers with an input on content-related aspects may be able to describe the potential for conceptual focus more precisely and do not remain on a surface-level, which strengthens Baumert and Kunters' (2006, p. 496) assertion that "content is the basis on which subject-specific didactic flexibility can develop."

This ends up in consequences regarding the content quality of prospective teachers' development programmes. Input on content-related aspects in case of the scalar product could help prospective teachers to unfold underlying conceptual elements that are crucial for teaching, especially for selecting tasks. In some constellations, didactic assumptions can already be described as fruitful for teaching the scalar product due to the activated categories of conceptual understanding Nevertheless, not all prospective teachers who received input automatically activate didactic categories in a targeted manner. For this reason, professional development programmes that deal with the mathematical content of scalar product must also develop content-related didactic categories. Further research is needed to specify the inductive categories that prospective teachers activate when selecting tasks, as well as to find deductive didactic categories that are compatible with condensing and unfolding elements related to the model of conceptual understanding for the scalar product.

References

Baumert, J., & Kunter, M. (2006). *Stichwort: Professionelle Kompetenz von Lehrkräften*. ZfE, 9, 469-520.

Boston, M., & Smith, M. (2011). *A 'task-centric approach' to professional development: Enhancing and sustaining mathematics teachers' ability to implement cognitively challenging mathematical tasks.* ZMD 43(6), 965-997.

Dorier, J.-L. (2000). *On the teaching of linear algebra*. Kluwer academic publishers.

Drollinger-Vetter, B. (2011). *Verstehenselemente und strukturelle Klarheit*. Waxmann.

González, M. J., Gómez, P., & Pinzón, A. (2020). *Characterising lesson planning: a case study with mathematics teachers.* Teaching Education, 31(3), 260–278.

Herrmann, J. (2025). Verstehenstypen von Lehrkräften zum Konzept Skalarprodukt. To appear in: *Beiträge zum Mathematik-unterricht 2025*.

Herrmann, J., Hoffmann, M., & Wessel, L. (in prep.). *Mit Darstellungen verstehen – Konzept-verständnis durch Darstellungsvernetzung in der analytischen Geometrie.*

Hoffmann, M. et al. (in prep.). *Grundvorstellungen, Aspekte und Darstellungen zum euklidischen Skalarprodukt. Spezifikation und Strukturierung des Lerngegenstandes.*

Lotz, J. (2022). *enaktiv-ikonisch-symbolisch. Eine semiotisch basierte Präzisierung und deren unterrichtspraktische Konkretisierungen*. Saarbrücken.

Mayring, P. (2014). *Qualitative Content Analysis. Theoretical Foundation, Basic Procedures and Software Solution*. Klagenfurt.

Prediger, S. (2019). *Investigating and promoting teachers' expertise for language-responsive mathematics teaching.* Mathematics Education Research Journal, 31(4), 367–392.

Prediger, S., et al. (2022). *Five principles for high-quality mathematics teaching: Combining normative, epistemological, empirical, and pragmatic perspectives for specifying the content of professional development*. Frontiers in Education, 7.

Prediger, S., & Zindel, C. (2017): *Deepening prospective mathematics teachers' diagnostic judgements: Interplay of videos, focus questions and didactic categories*. JMD, 5(3), 222-242.