Low-achieving secondary students learn mathematical problem solving -A longitudinal, qualitative video study

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Mathematical problem solving is hard to learn and difficult to teach – esp. for classrooms including low-achieving (LA) students. We investigate what difficulties LA students encounter when solving problems and how their problem-solving behavior develops within a period of nine months. Within a broader project, three pairs of LA students have been videotaped four times over a period of nine months. For this contribution, the video data of the first and fourth collection were analyzed. To do so, each problem-solving process was coded using four different codings: episodes, heuristics, self-regulatory activities, and success. We identified three main difficulties that students encounter: basic knowledge, stagnated development of heuristics, and a lack of self-regulation.

Keywords: heuristics, problem solving, resources, strategy keys, strategy prompts.

Introduction and theoretical background

In many countries around the world (as in Germany), becoming competent in solving problems is an integral goal of mathematics curricula (Liljedahl & Cai, 2021). Students are supposed to acquire strategies, apply those flexibly and, thus, master solving problems (NCTM, 2000). Yet, problem solving itself is a very complex process which is hard to learn (Schoenfeld, 1985). This applies especially for low-achieving (LA) students and students with learning difficulties (LD) (Montague et al., 2011). The situation in schools becomes even more delicate as teachers report that "teaching large heterogeneous groups of students with varied academic level" is stressful (Shernoff et al., 2011, p. 65). Again, if students show difficulties within "regular" mathematics lessons, they can be expected to show even more differences when working on non-routine problems.

Our review of the literature reveals a lack of studies regarding problem solving of LA students and students with LD with the exception of word problems for which it is known that "[s]tudents with LD are characteristically poor problem solvers. They typically lack knowledge of problem-solving processes, particularly those necessary for representing problems and, therefore, need to be taught those processes explicitly" (Montague et al., 2011, p. 263). At the same time, these students lack proper strategy use; instead, they "infrequently abandon and replace ineffective strategies, rarely adapt previously learned strategies, and typically do not generalize strategy use across domains" (Montague et al., 2011, p. 263). Yet, word problems often do not initiate non-routine problem-solving processes (Liljedahl, 2021, p. 25).

Thus, if it is that difficult for LA students and students with LD to master word problems, it will be even more difficult for them to work on non-routine problems. At the same time, these students should learn to solve problems as this is an essential skill for learning in the 21st century.

Research interest and questions

Here, we refer to problem solving as working on *non-routine* problems. We report about a longitudinal, qualitative study that addresses the learning of mathematical problem solving within regular math lessons including LA students, that is students who "experience mild but persistent learning difficulties" (Geary et al., 2012, p. 206). We investigate the following questions: (1) What difficulties do LA students encounter when solving mathematical problems? and (2) How does their problem-solving behavior develop over a period of nine months?

Design of the study

We conducted an intervention study involving 162 students (aged 11 to 14 years, grades 5 to 7) in Germany. All students are from a regular comprehensive school with all ability levels; hence, there is a representative selection of the early secondary students – also inclusive and integrative students. We gathered the students' math and German grades and used these to group the students into high, middle, and low achievers. We asked their teachers to verify this grouping based on their experience. A standardized maths test was not carried out within this study.

The design of our study is based on three current and well-established approaches for supporting teachers and students within very heterogeneous classrooms: (1) "allow students to work on mathematical problem solving tasks in pairs or in small groups", (2) "provid[e] all students with high-quality mathematics instruction [by] present[ing] the whole class with the same low-floor, high-ceiling, open-ended, problem-solving task", and (3) "teach mathematics through challenging task, and use enabling and extending prompts to differentiate instruction" (Russo et al., 2020, p. 49).

Over a period of nine months, the students encountered mathematical problems on a regular basis in their math lessons (approach 2). This is not only part of the national curriculum but also part of the intervention. To support the students, they were introduced to so-called *strategy keys* – these are cards in the shape of keys with one general heuristic written on each key, used to trigger the use of heuristics (approach 3: strategy prompts). We refer to heuristics as problem-solving strategies that students can implement when working on a problem, including "mental operations typically useful for the solution of problems" (Pólya 1945, p. 2). That strategy keys are effective in activating the use of heuristics and fostering self-regulatory activities has been shown empirically by Herold-Blasius (2021) in a project with mathematically interested 3rd and 4th graders. In the study at hand, the keys were introduced to the students within one introductory lesson; thereafter, they were allowed to use the keys whenever they "got stuck" within their problem-solving processes.

Within this project, qualitative data was collected at four different points of time. Thus, 30 pairs of students (approach 1) were videotaped and asked to think aloud while working on non-routine mathematical problems. After finishing the problem-solving processes, they were interviewed and their solution sheets were gathered. The chosen problems were structurally similar throughout the entire study, mainly arithmetic and sometimes geometric. This way, we aimed at comparing the processes more objectively than by using different problems each time.

Due to different reasons only eight of those 30 pairs were constant in their composition at all points of data collection. For the contribution at hand, we filtered the three low-achieving pairs of students.

Their problem-solving processes of the first and last video collection as well as the matching worksheets are focused on in this paper. Using this data, we illustrate how the encountered difficulties and if problem-solving skills develop in 9 months.

Two mathematical problems of the study are used as basis for the paper at hand:

- *Seven Gates*: A man picks apples. On his way to town, he has to pass seven gates. At each gate stands a guardian claiming half of the apples and one apple extra. At the end, the man has only one apple left. How many apples did he have at the beginning? (Bruder et al., 2005)
- *Traffic Jam in Space*: In order to reduce the amount of traffic between galaxies in space, the Intergalactic Council has decided to impose customs duties on commercial travelers. Each time a trader changes galaxies, he must now give up a quarter of his goods, plus one more piece. Plutix distributes toxic-green paint. Arriving at his destination, he still has 23 buckets after having to stop at the intergalactic customs three times and hand in goods. How many buckets did he have loaded at the beginning of his journey? (Collet, 2009)

Both problems are similar in their construction: Someone has to pass a certain number of steps (gates or duties) at which part of an initial amount of something has to be paid – either half or a quarter and one extra. In both problems, the amount at the end is given, so working backwards is the most appropriate strategy. However, other heuristics can either support the working backwards (e.g., aid to systemize or a figure) or open other approaches (e.g., working forward).

Methods – Coding of the data

Each problem-solving process is coded in four different ways in order to describe the problem-solving process as detailed as possible: 1) episodes within the process, 2) heuristics used in the process, 3) self-regulatory activities within the process, and 4) success of the process. Codings 1), 2), and 3) are process-oriented (i.e., using video data), whereas coding 4) is product-oriented (i.e., using the students' written worksheets). Each coding will be explained in the subsequent sections.

Process coding – Episodes: Schoenfeld (1985) defined an episode as period of time during which an individual does the same. He identified six types of episodes (see Table 1 for types relevant here).

Process coding – Heuristics: Appearances of heuristic techniques, such as drawing a figure, generating examples, or trying systematically, were coded using a manual based on ideas by Koichu, Berman, and Moore (2007) (cf. Rott, 2013). Codes relevant for this paper are specified in Table 2.

Process coding – Self-regulation: To code self-regulatory activities within problem-solving processes, we used Schoenfeld's (1985) differentiation between local and meta assessment. "*Local assessment* is an evaluation of the current state of the solution at a microscopic level". (Schoenfeld, 1985, p. 299) So, whenever students check their results, local assessment occurs within a problem-solving process, (see Table 3). *Meta assessment* did not occur within the given data and will not be addressed any further here. Additionally, we identified self-regulatory activities is the *interaction with the strategy keys* (cf. Herold-Blasius 2021, showing that using keys can be interpreted as an act of self-regulation). This is because students realize that they need help when they get stuck and they know where to get this help from – the strategy keys. Also, several pre-, while- and post-actional activities of self-regulation could be identified when interacting with the keys (Zimmerman, 2000).

Code	Description
Reading or rereading	The problem is read or re-read. It also contains the silent period after reading.
Analyzing the problem	The problem solver tries to understand the problem, e. g. by thinking about what is sought and what is given.
Exploring	The problem solver investigates the problem which might be more or less chaotic.

Table 1: Codes to identify episodes

Code (Abbreviation)	Description
Sought and given (S & G)	Checking for information given in the problem and what is searched for
Aid for systematization (AfS)	Introducing structuring elements that help to carry out activities
Working backwards (WoBa)	Focusing the aim of the problem/the sought and working from there to the starting point; sometimes mathematical operations must be reversed

Table 2: Codes to identify heuristics

Code (Appreviation)	Description
Local assessment	Evaluation of the current state of the solution, e. g. check results
Interaction with strategy keys	Students read, touch, discuss about the keys or choose a specific key

Table 3: Codes to identify self-regulatory activities

Product coding: Solution

The students' solution sheets, thus, products, were graded in four categories of success:

1. *No access*, when they showed no signs of understanding the task properly or did not work on it meaningfully.
2. *Basic access*, when the pupils mainly understood the problem and showed a basic approach.
3. *Advanced access*, when they understood the problem properly and solved it for the most part.
4. *Full access*, when the pupils solved the task properly and presented appropriate reasons, if necessary. (Rott, 2013, pp. 115–116)

Results

Here, we present the problem-solving processes of three pairs of LA students that worked on the problems Seven Gates and Traffic Jam in Space (1st and 4th video recording, resp.).

K & M: Seven Gates (1st video collection)

During K & M's work on the problem Seven Gates, they spent most of their time in exploration phases. This indicates that both girls did not follow a specific plan.

Concerning heuristics, M & K *reread* the Seven Gate problem and applied solution methods as if the problem were a *routine task*. This means they used all given numbers and put them in some (possibly random) mathematical relation:

M: Can we not just do seven times seven because it's seven gates and then plus seven because another apple for each? [writing: $7 \cdot 7 = 49; 49 + 7 = 56; 56 += 57;$ *"The man picked up 57 apples."*]

Self-regulatory activities could not be identified and their solution was coded as basic access.

Combining, it seems as if K & M did not understand the problem properly. They understood that there are seven gates and seven guardians and that there is one apple left. However, they could not bring this information together and, thus, treated the problem as if it were a routine task.

K & M: Traffic Jam in Space (4th video collection)

During the problem Traffic Jam in space, both worked within an exploration phase. Concerning their heuristics, they *reread* the problem – as they did in the Seven Gates problem. Yet, they also used an *aid to systemize* as they first drew and later numbered the gates (1, 2, 3). Also, they talked about how they want to proceed and decided that *working backwards* to be an expedient approach. Again, self-regulatory activities could not be identified and their solution was coded as basic access.

Concluding, K& M understand the problem much better, talk about possible heuristics to use and find helpful mathematical operations to work on the problem. Still, to solve the problem correctly they need knowledge about fractions – which they should have but did not show or seem to have. So, when solving mathematical problems, it is not sufficient to understand the problem and to select an appropriate heuristic. It is also necessary to bring along adequate mathematical knowledge.

M & F: Seven Gates (1st video collection)

M & F read the problem and analyze it. M has an idea to calculate straight away (implementation):

M: There is one [apple] left. If they. Double of one is two. Then an extra apple is three.
F: That's the second gate.
M: If we had six and one it would be seven. So, we had two gates.

M has an idea; F follows it and both carry out their plan. Concerning their heuristics, they *work backwards* and use an *aid to systemize* the number of gates by numbering them. At half of the process, F checked the results and figured that doubling the number of apples and then adding one does not work when working forward. However, this self-regulatory activity does not lead to a change of calculation. Still, with 255 apples as their final result, they have a solution coded as advanced access.

Summing up, M & F talk with and understand each other. It seems to be sufficient if one of them has an idea because the other one follows. They are able to select appropriate heuristics and even show a self-regulatory activity. Yet, only recognizing that something might be wrong does not make them change their solution path. Hence, they might be skilled with a seed of self-regulation, that might not have been developed fully yet (Herold-Blasius, 2021).

M & F: Traffic Jam in Space (4th video collection)

After reading this problem, both recognize that they have worked on a similar problem before. This might be one reason why M decides to *work backwards* and to draw the customs stations (*aid for systematization*) right at the beginning. Their first idea is to add one to 23 and then to add one quarter of 24 to 24 which is 30. However, they check their results (self-regulatory activity) and recognize that their calculation does not work – having 30 and taking away one quarter and one extra. At some point (after 11 min) M decides to just do the calculations – without thinking about it anymore. At the end (after 17 min), they explain their strategy use explicitly:

M:	I have calculated 62.5 divided by 4.
Interviewer:	62.5 divided by 4. And why?
M:	Because when you work backwards you can also do it forwards. This way you can check it.

To conclude, both are able to select heuristics and to check their result (self-regulatory activity) – in both processes. Yet, they cannot use this to change their procedure and do not find the correct result. Within the nine months, they did not learn how to change their working backwards in a productive way. Possibly, explicit teaching could have helped here to develop this self-regulatory seed.

F & N: Seven Gates (1st video collection)

F & N start their work by analyzing the problem and roughly estimating a possible result.

N:	There must be an awful lot of them.

They then try to solve the problem within an exploration phase. In terms of heuristics, they looked for the *sought and given* first and *reread* the problem after about five minutes. Shortly after, they interacted with the strategy keys (self-regulatory activity) which led them to a new strategy – *working backwards*. Later they combined this heuristic with an *aid to systematize* to keep track of the number of gates. Their entire process took 14 minutes and F & N's solution was coded as advanced access.

In F & N's process it was noticeable that they extensively discussed the order of operations – first addition and then multiplication or vice versa. At the end, the writer determined the order. Also, after interacting with the strategy keys, they employed the heuristic *work backwards*.

F & N: Traffic Jam in Space (4th video collection)

After reading the problem they analyze it and then work on it. Concerning heuristics, they *reread* the problem, looked for the *sought and given* and used an *auxiliary element* by marking relevant information in the given text. After 7 minutes, they started *working backwards* and later drew some *figures* visualizing the quarter. After 34 minutes, they again used an *aid to systematize* to keep track of the number of gates. Their process lasted ca. 42 minutes and their solution was coded as advanced access.

Both students interact with the strategy keys at three points within the process (self-regulatory activity). However, they did not find a productive way of integrating them.

F & N's problem-solving process is characterized by showing seeds of purposeful problem-solving behavior: First, they mark relevant information but then do not really use it within their solution process. Second, they talk about the order of operations; yet, F – the writer – determines it without N arguing against it. Third, they use circles as visualization of the quarters. Still, this is not used for their calculations or any progress within the problem-solving process. It is remarkable that both calculate with approximated numbers. So, it seems as if their conception of numbers and fractions is suitable for everyday use. Hence, their mathematical knowledge seems to be at a sufficient level.

In conclusion, F & N show similar behavior e.g. by working backwards (key: work backwards). At the same time, they use new behavior such as marking information (key: use different colours) or drawing figures (key: draw a picture). Possibly, both students have internalized the strategy keys and can now employ heuristics without even looking at the keys. They have expanded their heuristics repertoire and internalized heuristics, although this does not lead them to correct solutions yet.

Discussion

For this contribution, we observed the problem-solving behavior of three pairs of LA students over a period of nine months. We showed what kind of difficulties LA students encounter when solving mathematical problems and how their problem-solving behavior develops over the given period.

Three obstacles have been identified: lack of knowledge, stagnated development of heuristics, and lack of self-regulation. First, students (see K & M) need a certain level of mathematical knowledge to properly engage in problem solving. This result is not surprising as Schoenfeld (1995) had already stated this. Yet, there is no concept on how a lack of mathematical knowledge could be compensated using aids without revealing too much – and this is crucial especially for LA students. Students need sufficient knowledge or to develop "the ability to retrieve basic facts from long-term memory" (Geary et al., 2012, p. 206) in order to experience a non-routine problem-solving process.

Second, the strategy keys aimed at activating known heuristics. F & N know heuristics and are able to select appropriate ones. Still, this is not sufficient to successfully work on mathematical problems, as they do not find the correct result. Their heuristic development has stagnated within our study.

Third, F & N interacted with the strategy keys and M & F checked their results within the problem-solving process. Thus, both pairs showed self-regulatory activities. Yet, this did not result in a correct solution. We claim that regarding self-regulation, these students dispose of only seeds rather than a full self-regulatory competence (Stein, 1995) – so they know a self-regulatory activity and they know how to use it, but they cannot integrate it in their processes in a goal-oriented manner.

In the future, it should be investigated how LA students can be enabled to successfully solve mathematical problems. Possibly the introduction phase of a problem solving lesson could be a fruitful resource to support especially LA students (see Häsel-Weide & Nührenbörger, 2022).

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