CENTER FOR TAX AND ACCOUNTING RESEARCH

Center for Risk Management

TAXATION, ACCOUNTING, AND FINANCE

TAF WORKING PAPER

No. 104 / May 2026

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Abstract

Volatility modeling is utilized across numerous fields including finance, environmental studies, and social sciences. It is particularly relevant in scenarios where understanding and predicting conditional variability is crucial, such as when dealing with incremental or time-dependent data. In this paper, novel short and long memory volatility models of the EGARCH family are introduced and analyzed, which are closely related to the well-established EGARCH model proposed by Nelson (1991) but share desirable theoretical properties in several dimensions. Recently developed members of the so-called EGARCH family, which introduces a modulus-log transformation proposed by John and Draper (1980) and a power transformation for the size and magnitude effect to tackle the problem with near-zero innovations and the asymmetric impact of positive and negative shocks on the volatility, are discussed. After a theoretical discussion of the proposed and related volatility models, the practical performance of the elaborated volatility models is compared to well-established and traditional GARCH approaches. A general QMLE algorithm is proposed to estimate the model parameters. The practical relevance of the advanced models is illustrated through a comparative study. By applying these volatility models to a variety of international stock index returns, this paper identifies market-specific characteristics as well as unique strengths and weaknesses of discussed volatility models. Although the practical performance of the recently introduced models is comparable to those obtained by the traditional EGARCH model, they generally outperform traditional non-exponential volatility models used as benchmarks and thus provide a useful alternative to existing short and long memory volatility models.

Keywords: Modulus Log-GARCH, Modified (FI)EGARCH, Modulus asymmetric (FI)Log-GARCH, (FI)EGARCH, long memory, modulus-log transformation, QMLE, model selection, implementation in R

1 Introduction

Playing a crucial role in both academic research and practical applications across various fields, econometric modeling and forecasting of volatility has been among the most important developments in empirical finance, promoted by the tremendous growth of financial derivatives, quantitative trading and risk modelling. As more accurate volatility models enable better decision-making, compliance with regulation as well as strategic planning, and contributes to overall economic stability and growth, various different volatility models have been developed, aiming at overcoming the weaknesses and problems of previously introduced representations.

The volatile nature of the financial markets poses an ongoing challenge in quantitative risk management when modeling financial risks. In particular, exogenous shocks such as the global financial crisis in 2008 and the COVID-19 pandemic have a considerable impact on market dynamics, liquidity, volatility and investor sentiment. Precise risk measures play an important role in managing these risks. As financial markets become more integrated and financial liberalization continues to shape the structure and behavior of financial markets, investors are faced with highly intricate market structures involving not only complex information flows but also highly volatile asset prices driven by (mostly rational) expectations. As models like the signaling theory (Mussa, 1981) and microstructure research (Bacchetta and Wincoop, 2008) indicate the crucial roles of expectations, information dissemination, and the error-correcting behavior of speculative investors, these factors are recognized as key drivers of price fluctuations and market volatility. Financial time series are therefore inherently characterized by the occurrence of so-called volatility clusters - persistent periods of high or low variance (Bera and Higgins, 1993). GARCH-type models, known for their precision and efficiency in dynamically assessing variance processes, offer the ability to capture and forecast the dynamic and time-varying behavior of conditional volatility, which makes them indispensable tools in finance as they help financial institutions and private investors to maintain adequate capital reserves and mitigate potential losses. The applications span the evaluation of primary financial assets (Merton, 1980) and derivatives (Wiggins, 1987), the analysis of information flows (Conrad et al., 1991), the development of optimal hedging strategies (Baillie and Myers, 1989), event studies (Connolly, 1989), and the assessment of both short- and long-term macroeconomic fundamentals using term structure theories (e.g., Cox et al., 1981).

Since the introduction of the autoregressive conditional heteroskedasticity (ARCH) model by Engle (1982), numerous parametric methods for dynamically modeling high-frequency financial market volatility in return series have been developed to account for diverse asset characteristics and informational properties, enhancing the accuracy of market risk assessment and forecasts. Among the most prominent are the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model introduced by Bollerslev (1986) and the Exponential GARCH (EGARCH) model developed by Nelson (1991). These models can be interpreted as ARMA-type processes applied to the conditional second-order moments and the logarithm of the conditional variance, respectively. While a consistent empirical observation across applications of those models is the pronounced persistence in the estimated conditional variance dynamics (Bollerslev et al., 1992), simple short memory GARCH-type models share the similarity of implying exponentially decaying autocorrelations of the squared innovation series and thus cannot control for long memory in the conditional dynamics.

Recently, analogous to the extension of the ARMA to the FARIMA introduced by Granger and Joyeux (1980), modeling long memory in volatility is a crucial area in financial econometrics and internal risk management, as traditional models reach their limits due to the increasing complexity of the financial market. Prominent models that address long memory in volatility include the FIGARCH (fractionally integrated GARCH, Baillie et al., 1996), the FIEGARCH (fractionally integrated EGARCH; Bollerslev and Mikkelsen, 1996), and the FIAPARCH (fractionally integrated asymmetric power ARCH, proposed by Ding et al., 1993; Tse, 1998). More recently, Feng et al. (2020) expanded the Log-GARCH model (logarithmic GARCH, initially developed by Geweke, 1986; Pantula, 1986; Francq et al., 2013) to the FILog-GARCH. Both FIEGARCH and FILog-GARCH are specific instances of exponential volatility models (Zaffaroni, 2009) which will be the subject of the paper at hand. Empirical research documents that long memory GARCH models excel in accurately predicting and frequently outperform traditional short memory GARCH models in effectively modeling conditional volatility (Giot and Laurent, 2003; Degiannakis, 2004; Tang and Shieh, 2006; Grané and Veiga, 2008; Härdle and Mungo, 2008; Baillie and Morana, 2009; Demiralay and Ulusoy, 2014; Aloui and Ben Hamida, 2015; and Royer, 2022). However, since there is no final consensus on which models performs best in modeling conditional volatility, the paper at hand performs a large empirical comparative study which highlights unique strengths and weaknesses of the theoretically introduced and discussed models.

This paper primarily focuses on exponential volatility models, due to their desirable statistical properties, their ability to model asymmetric shocks to the volatility process and their capacity to maintain finite fourth moments (across any level of long memory). While the (FI)Log-GARCH model demonstrates limitations in performance because it does not account for asymmetry effects and struggles with handling (nearly) zero observations, the (FI)EGARCH produces finite moments only if the innovation distribution possesses a moment generating function (mgf), which excludes commonly used innovations, such as those following a t-distribution. To address these challenges, we propose the use of new parametric volatility models which are part of a general EGARCH family. The EGARCH family provides a general framework governing the traditional exponential and logarithmic volatility models, which can be adjusted to enhance practical performance and overcome statistical weaknesses. Even though the framework allows for countless variants, the following focuses on two proven volatility models (with its short and long memory variants) that are intended to underline the adaptability, flexibility, and the power of the newly introduced setting with its broad coverage. These models incorporate an asymmetric term, remain unaffected by zero observations, and can achieve finite moments up to a specified order, provided that the innovations possess finite moments to the necessary degree. Using quasi-maximum-likelihood estimation (QMLE), this paper is able to reveal specific model behaviors of these newly introduced members by comparing the coefficients under the most widely used conditional normality assumption, testing them against traditional benchmark models. The statistical description and the enhanced comparative empirical study that indicates good performance of the novel volatility model framework is the central contribution of this paper.

To facilitate practical application, R functions tailored for modeling returns with both short and long memory in volatility were developed. To test the different models' performance, the volatility models are applied to return series of prominent stock market indices. Overall, the comprehensive, comparative empirical study indicates that the proposed models demonstrate an improved performance over symmetric and widely used non-exponential models. While the advanced volatility models perform comparably to traditional exponential models, they form an attractive alternative containing theoretical advantages. Additionally, the findings offer corroborative evidence that long-term volatility is a time-varying, oscillatory process. The net effect on short-term conditional volatility is contingent upon both the direction and magnitude of shocks. Comparative analysis of the estimated GARCH coefficients reveals variations in exposure, shock intensity, and the persistence of short-term volatility. Furthermore, findings indicate that instances where $d \geq 0.5$ are typically linked to structural breaks and gradually shifting scales rather than stochastic influences. Once potential deterministic non-stationarity is accounted for, the error process should be stationary with finite fourth moments.

The paper is organized as follows: Section 2 briefly summarizes the most popular short and long memory GARCH models that are related to the more sophisticated models addressed and introduced in the following. These models are used as benchmarks in the subsequent study. The general concept of the double power modulus EGARCH as the general framework of the overarching broad EGARCH family is defined in Section 3. Therein, as two special cases of the general framework, the modified EGARCH and the recently introduced modulus adjusted Log-GARCH are depicted. Section 4 describes the estimation procedure before Section 5 addresses the comparative study and Section 6 concludes.

2 Existing related approaches

Understanding the limitations of simpler models - such as their inability to account for leverage effects and their incapability to cope with observations close to zero - motivates the development of enhanced versions, which address these shortcomings. Well-established GARCH models, which capture key features of financial time series and provide the fundamental intuition needed to understand more sophisticated extensions, serve as benchmarks, allowing for robust empirical comparisons when determining the appropriateness of more sophisticated models in capturing the unique characteristics of financial data by minimizing prediction errors and improving reliability.

Emerging as special cases of the general EGARCH-class defined in Nelson (1991), the proposed models are part of the general EGARCH family with special adaptations to improve general applicability, performance and statistical properties. Nevertheless the existence of strictly stationary solutions for the models can be obtained by Theorem 2.1 in Nelson (1991). Weak stationarity and existence of finite moments of the traditional and novel GARCH models will be discussed briefly. Theoretical properties of existing models and the anticipated results of the newly introduced models will be presented without formal proof, as an in-depth theoretical analysis of these models falls outside the scope of this paper.

2.1 Short versus long memory dependencies

Comparing short- and long memory versions of GARCH models is crucial as different model specifications are capable of modeling different volatility dynamics and persistence patterns, which is essential for making informed financial decisions and is vital for accurate risk management, pricing of financial instruments (mainly options), portfolio allocation and optimization, effectively hedging potentially materializing risks, and understanding market dynamics. The short and long memory properties of GARCH models refer to their ability to model divergent persistence characteristics in the volatility. Traditional short memory parametric (G)ARCH models imply exponentially decaying autocorrelations of squared innovations, which induces the volatility process to decay relatively quickly over time. While short memory models often have difficulties in accurately capturing the persistent patterns and long-term correlations in price movements, they are based on the assumption of a summable autocovariance function $\gamma(k)$. The long memory property refers to the persistence of shocks or the slow (hyperbolic) decay of autocorrelations over time, first documented by Ding et al. (1993). Afterwards, many researchers found evidence for the presence of long memory in (intraday and high-frequency) volatility series by analyzing empirical autocorrelations of absolute or squared returns (e.g. Ding and Granger, 1996; Andersen and Bollerslev, 1997; Andersen et al., 1999; Cotter, 2005). For a time series r_t with long memory in volatility, the autocorrelation function of r_t^2 satisfies:

$$\rho(k) \sim c_\rho k^{2d-1} \; as \; k \to \infty$$

where c_ρ is a constant, $\sim$ means that the ratio of both sides tends to 1 as $k \to \infty$, and d is the fractional integration parameter. For $d \in (0, 0.5)$, the autocorrelation function decays hyperbolically, indicating long memory. In this case, the process is long-term correlated and the absolute sum of autocovariances is not summable anymore. If the time series exhibits long memory properties, the assessment and forecast of future market fluctuations is less precise with traditional short memory models. To tackle this problem, some of the most important extensions are the fractionally autoregressive integrated moving average (FARIMA) model by Granger and Joyeux (1980), the fractionally integrated GARCH (FIGARCH) model by Baillie et al. (1996) and the long memory GARCH (LM-GARCH) model developed by Robinson and Henry (1999).

2.2 Generalized (long memory) Autoregressive Conditional Heteroskedasticity

A significant milestone in modeling conditional volatility processes was the development of the Autoregressive Conditional Heteroskedasticity (ARCH) model by Engle (1982), which is capable of modeling and predicting conditional heteroskedasticity and volatility clustering in financial time series.[1] Following Engle's framework the conditional variance of an asset depends on its past squared returns, while the unconditional variance is assumed to remain constant over time. Bollerslev (1986) augmented the basic ARCH and developed the generalized ARCH (GARCH) model, where additionally past values of the conditional variance are considered, which allows for a more flexible and parsimonious lag structure, reducing the number of parameters required (Bera and Higgins, 1993). Given a financial time series with logarithmic returns $r_t^*, t = 1, \dots, n$ and $E(r_t^*) = \mu_{r^*}$, the centralized returns are $r_t = r_t^* - \mu_{r^*}$ and the time-varying conditional variance σ_t^2 following a GARCH(p, q)-process is defined by:

$$r_t = \sigma_t \varepsilon_t \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{2}$$

where ε_t is a series of i.i.d. innovations with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2 - 1) = 0$. Let $\mathcal{F}_t = \sigma(\{r_t : s < t\})$ be a sigma algebra representing the history of the return process up to time t, so that $(\mathcal{F}_t)_{t \in \mathbb{Z}}$ is a natural filtration (McNeil et al., 2015). The random variable $var(r_t | \mathcal{F}_{t-1}) = \sigma_t^2$ is the conditional variance of r_t. Different parametric volatility models arise from diverse specification of σ_t^2. Here, the non-negativity conditions $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $\forall i \in \{1, \dots, p\}$, and $\beta_j \geq 0$ for $\forall j \in \{1, \dots, q\}$, with $p \geq 0$ and $q > 0$, are required. For $q = 0$, the process reduces to an ARCH(p) process, and for $p = q = 0$, it becomes a simple white noise process (Bollerslev, 1986). For strict stationarity of GARCH processes see e.g. Nelson (1990b), Bougerol and Picard (1992), and Ling and McAleer (2002). Assuming the GARCH process is stationary, a GARCH(1,1) process can be obtained by rewriting:

$$\sigma_t^2 = \frac{\alpha_0}{(1 - \beta_1 B)} + \alpha_1 r_{t-1}^2 + \alpha_1 \sum_{i=1}^{\infty} \beta_1^i r_{t-i-1}^2 \tag{3}$$

which is an ARCH(∞) model. Therefore the GARCH model constitutes an attractive specification for financial time series application because it provides a parsimonious representation of the memory characteristics usually observed in the variance of financial returns. By defining $\eta_t = r_t^2 - \sigma_t^2$, $\phi_i = \alpha_i + \beta_i$, and $\psi_j = -\beta_j$ it is easy to see that the GARCH takes on an ARMA(m, q) representation in r_t^2:

$$\phi(B) r_t^2 = \alpha_0 + \psi(B) \eta_t \tag{4}$$

where the process $\{\eta_t\}$ follows a martingale difference series and therefore a weak white-noise process and can be interpreted as the innovations of the conditional variance, because it has a zero mean and is

[1] The relevance to model volatility of an asset as a weighted sum of dependent squared returns in the form of an autoregression of (in-)finite lag p is evident, among other things, in the fact that Engle was awarded the Nobel Prize in Economics in 2003.

serially uncorrelated. $m = max\{p, q\}$, and $\phi(B) = 1 - \sum_{i=1}^{m} \phi_i B^i$ as well as $\psi(B) = 1 + \sum_{j=1}^{q} \psi_j B^i$ are the AR(m) and MA(q) polynomials, respectively. A GARCH process is covariance stationary if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = \hat{P} < 1$ (Bollerslev, 1986).[2] Financial time series tend to exhibit heavy tails, occurring when the fourth-order moment is infinite and the distribution has more extreme values than assumed by the normal distribution.[3] The existence of the fourth-order moment is a prerequisite in the estimation process. The tail behavior of stationary GARCH processes has been intensively investigated in the literature (see e.g. Bollerslev, 1986; Basrak et al., 2002; Karanasos, 1999; He and Teräsvirta, 1999).

To model long-term dependencies in the GARCH framework, Baillie et al. (1996) defined the fractionally integrated GARCH model. Due to the incorporation of a fractional differencing operator $(1 - B)^d$ in the AR polynomial, allowing a slow hyperbolic decay rate of the lagged quadratic innovations, the FIGARCH grants persistent time dependencies. The FIGARCH(p, d, q) is defined by:

$$\phi(B)(1 - B)^d r_t^2 = \alpha_0 + \psi(B)\eta_t \tag{5}$$

Here, d represents the long memory parameter. For statistical properties see Chung (2002) and Conrad and Karanasos (2006). The concept goes back to fractional Brownian motion (Hurst, 1951; Mandelbrot and Van Ness, 1968). The fractional difference operator can be defined as a hypergeometric function (Hosking, 1981; Granger and Joyeux, 1980):

$$(1 - B)^d = \sum_{k=0}^{\infty} (-1)^k \Gamma(d + 1)\Gamma(k + 1)^{-1}\, \Gamma(d - k + 1)^{-1} B^k \tag{6}$$

where $\Gamma(\cdot)$ denotes the gamma function. For $0 < d < 1$, this operator captures a slow, hyperbolic decay of past shocks, allowing the model to represent long memory behavior. For $d = 0$, the FIGARCH model is reduced to a GARCH model and for $d = 1$ to an IGARCH model introduced by Engle and Bollerslev (1986). If $0 < d < 0.5$, the process is stationary and exhibits long memory, meaning that shocks to volatility decay slowly over time. If $0.5 \leq d < 1$, the process is non-stationary but mean-reverting.[4]

Another representation of the FIGARCH model is the ARCH(∞) notation, analogous to the FARIMA model. Assume that all roots $\phi(B)$ and $\psi(B)$ lie outside the unit circle, then:

$$\psi(B)\sigma_t^2 = \alpha_0 + (\psi(B) - \phi(B)(1 - B)^d) r_t^2 \tag{7}$$

$$\begin{aligned} \sigma_t^2 &= \tilde{\omega} + \left(1 - \psi^{-1}(B)\phi(B)(1 - B)^d\right) r_t^2 \\ &= \tilde{\omega} + \lambda(B) r_t^2 \end{aligned} \tag{8}$$

[2] The persistence parameter $\hat{P}$ determines the persistent duration of a trend or shock. If $\hat{P} = 1$, there is no mean reversion effect in the conditional variance and the unconditional variance is undefined.

[3] In a GARCH(1, 1) model with $\varepsilon_t \sim N(0, 1)$, a finite fourth-order moment exists when $3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$. The kurtosis can then be determined as follows $E(r_t^4) = \frac{3\alpha_0^2(1+\alpha_1+\beta_1)}{(1-\alpha_1-\beta_1)(1-\beta_1^2-2\alpha_1\beta_1-3\alpha_1^2)}$. If a value greater than 3 is present, the kurtosis is considered leptokurtic and the distribution is heavy tailed. If the kurtosis takes on a lower value, the distribution is more evenly spread.

[4] In the context of linear processes, values of $d \in (-0.5, 0)$ are occasionally considered. However, such negative values of d are not pertinent in volatility modeling and will therefore be disregarded in the present analysis.

with $\lambda(B) = \lambda_1 B + \lambda_2 B^2 + \cdots$ and $\tilde{\omega} = \alpha_0 \psi(B)^{-1}$. The non-negativity condition $\lambda_k \geq 0$ must apply for $k = 1, 2, \ldots$. The coefficients of r_{t-i}^2 in the FIGARCH, say λ_i, are of the form $\lambda_i \sim Ci^{-d-1}$, indicating that r_t^2 exhibits long memory.

The theoretical properties of the FIGARCH model have been rigorously investigated in the literature. Baillie et al. (1996) and Bollerslev and Mikkelsen (1996) established conditions to ensure the non-negativity of the conditional variance, specifically for models up to first-order. Conrad and Haag (2006), building upon the framework of Nelson and Cao (1992), generalized these findings to accommodate higher-order models (with $q \leq 2$).[5] While the FIGARCH model is not covariance stationary, Baillie et al. (1996) provided arguments in favor of its strict stationarity and ergodicity for $d \in [0,1]$, employing a dominance-based reasoning analogous to Nelson (1990) and Bougerol and Picard (1992) in the context of IGARCH models. Further advancements were made by Douc et al. (2008), who demonstrated that, under additional assumptions regarding the distribution of the innovation process ε_t, the FIGARCH(0, d, 0) process admits a unique causal stationary solution satisfying $E(|r_t|^{2c}) < \infty$ for all $c < 1$. More recently, Giraitis et al. (2018) resolved a longstanding conjecture posed by Ding and Granger (1996), deriving necessary and sufficient conditions for the existence of a covariance-stationary solution to the FIGARCH process defined by $\sigma_t^2 = \omega + (1 - (1 - B)^d) r_t^2$, under the restriction $\omega = 0$ and $d \in (0, 0.5)$. However, the stationarity for $E(r_t^2) = \infty$ still remains an open question. While the unconditional variance does not exist if $d > 0$, Karanasos et al. (2004) and Robinson (1991) defined the FIGARCH in terms of $r_t^2 - \sigma^2$ and introduced the covariance-stationary so-called LM-GARCH models.

The theoretical foundation of the GARCH framework has been significantly strengthened by a range of model extensions designed to address key stylized facts observed in financial time series. These include: (i) the presence of leverage effects (Black, 1976); (ii) persistent volatility clustering (Ding et al., 1993; Ding and Granger, 1996); and (iii) the phenomenon of time-varying unconditional variance (Feng, 2004; Mikosch and Stărică, 2004).

2.3 (Long Memory) Asymmetric Power GARCH

By imposing a Box-Cox power transformation to the volatility process and the asymmetric absolute residuals, Ding et al. (1993) introduced the asymmetric power GARCH (APARCH) to tackle the problem with asymmetric responses to positive and negative shocks (leverage effect) initially discovered by Black (1976).[6] The APARCH model can be represented as follows:

[5] Notably, their analysis demonstrates that the conditional variance sequence $\{\sigma_t^2\}$ can remain almost surely non-negative even in cases where parameters such as ϕ_i and ψ_i are negative. In contrast, they also revealed that negative conditional variances may occur despite all model parameters being strictly positive, thereby emphasizing the importance of explicitly verifying non-negativity constraints in empirical implementations.

[6] Black (1976) observed that financial markets react more strongly to negative information than positive and that volatility increases with bad, eg. when returns are lower than expected, and decreases with good news (Nelson, 1991; Zaffaroni 2009).

$$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \phi_i(|r_{t-i}| - \gamma_i r_{t-i})^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta \tag{9}$$

where $\omega > 0$ is the baseline level of volatility, $\phi_i \geq 0,\ i = 1, \ldots, p$ are real-valued coefficients and measure the impact of previous period's shock on the current volatility in a non-linear way, $\beta_j \geq 0, j = 1, \ldots, q$ measures the persistence in volatility, $|\gamma_i| \leq 1, i = 1, \ldots, p$ is the asymmetry parameter, and $\delta \geq 0$ measures the degree of Box-Cox power transformation applied to the volatility process, giving greater weight to outliers and therefore accounting for specifications controlling for non-normal data. The leverage effect is expressed by γ_i, which takes a positive value if a leverage effect exists. Through the estimation of the optimal power term δ, different model specifications are possible, and the assumption of a specific distribution for model estimation is not a prerequisite (McKenzie and Mitchell, 2002).[7] The stationarity condition is $\sum_{i=1}^{p} \phi_i + \sum_{j=1}^{q} \beta_j < 1$. Otherwise, the unconditional variance does not exist, and the asymptotic properties of the maximum likelihood estimators are uncertain (Bollerslev, 1986).

The long memory extension of this model was provided by Tse (1998) as well as McCurdy and Michaud (1996). By generalizing a variety of models, the FIAPARCH arises as a versatile and highly flexible tool for capturing the complex dynamics of financial volatility. It is defined as follows:

$$\begin{aligned} \sigma_t^\delta &= \omega + \left(1 - \beta^{-1}(B)\phi(B)(1-B)^d\right)(|r_t| - \gamma_i r_t)^\delta \\ &= \omega + \lambda(B)\,(|r_t| - \gamma_i r_t)^\delta \end{aligned} \tag{10}$$

where $\omega = \frac{\alpha_0}{1-\beta(1)}$ and $\lambda(B) = \sum_{i=1}^{\infty} \lambda_i B = \left(1 - \beta^{-1}(B)\phi(B)(1-B)^d\right)$, $\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j$, where β_j, $j = 1, 2, \ldots, q$, are real-valued coefficients, and $\phi(B) = 1 - \sum_{j=1}^{q} \phi_i B^i$, where ϕ_i, $i = 1, 2, \ldots, p$, are also real-valued coefficients and the remaining parameters align with the short memory pendant. By defining $\eta_t = f(r_t) - \sigma_t^\delta$, $f(r_t) = (|r_t| - \gamma_i r_t)^\delta$ and $\alpha_i = \phi_i + \beta_i$ it is easy to see that the FIAPARCH takes on an ARMA representation:

$$\alpha(B)(1-B)^d f(r_t) = \alpha_0 + \beta(B)\eta_t \tag{11}$$

Capable of capturing persistence in volatility, another model, called the ARCH(∞), was defined by Robinson (1991) and further analyzed by Giraitis et al. (2000), Kazakevicius and Leipus (2002) as well as Douc et al. (2008). More recently, an ARCH(∞) extension of the APARCH that accounts for conditional asymmetry in the presence of severe long memory was proposed by Royer (2022), which nests the ARCH(∞) as well as the Threshold-ARCH(∞) (Bardet and Wintenberger, 2009). To guarantee the non-negativity of the infinite-order coefficient series $\left(1 - \beta^{-1}(B)\phi(B)(1-B)^d\right)$, and thereby in combination with the condition $\omega > 0$ ensuring that the resulting conditional variances remain strictly

[7] Particularly for a leptokurtic distribution, results are superior to the GARCH model, which assumes a normal distribution and therefore δ is pre-set to 2 (McKenzie and Mitchell, 2002). In the APARCH, the distribution of returns must be determined by higher-order moments, e.g. skewness and kurtosis, which often results in a power term that is significantly different from zero, one or two (Hentschel, 1995; Ding et al., 1993).

positive, the optimization procedure incorporates the sufficient conditions outlined in Footnote 6 of Tse (1998), originally derived by Bollerslev and Mikkelsen (1996).

The APARCH model governs several other models as special cases, which makes the APARCH a generalized framework that can be adapted to fit different types of data. These include the ARCH (Engle, 1982), the GARCH (Bollerslev, 1986), the TS-GARCH (Taylor, 1986; Schwert, 1989), the GJR-GARCH (Glosten et al., 1993), the TARCH (Zakoian, 1994), and the Log-ARCH (Geweke, 1986).

2.4 (Long memory) Exponential GARCH

In the following, models of the so called EGARCH family are proposed and analyzed. The wide class of exponential GARCH models was initially introduced by Nelson (1991). In contrast to Bollerslev's model, it is capable of modeling the asymmetric impact of positive and negative shocks, gives greater weight to negative shocks and hence accounts for the leverage effect. The EGARCH is defined as:

$$\ln \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i\big(\kappa\varepsilon_{t-i} + \gamma(|\varepsilon_{t-i}| - E|\varepsilon_{t-i}|)\big) + \sum_{j=1}^{q} \beta_j \ln \sigma_{t-j}^2 \tag{12}$$

$$= \omega + \phi^{-1}(B)\psi(B)g_E(\varepsilon_{t-1}) \tag{13}$$

where α_0 expresses the constant term that controls for the long-run volatility level, the β_j parameters capture the persistence of volatility, κ represents the asymmetric impact of shocks on volatility, $\phi(z)$ and $\psi(z)$ are corresponding characteristic polynomials of order p and $q-1$, which are expected to have no common factors, all roots of $\psi(B)$ lie outside the unit circle, and $g_E(\varepsilon_t) = \kappa\varepsilon_t + \gamma[|\varepsilon_t| - E(|\varepsilon_t|)]$ is a non-degenerate, strictly and weakly stationary, ergodic i.i.d. zero-mean error process with finite variance, called the news impact function (see Nelson, 1991 and Bollerslev and Mikkelsen, 1996). κ describes the sign effect of shocks on the volatility process, captures the leverage effect and is typically negative (Nelson, 1991). If $\kappa = 0$, positive and negative shocks of the same magnitude affect volatility equally. If $-1 \leq \kappa < 0$, negative shocks are more heavily weighted than positive ones. γ represents the magnitude effect of the shock and is usually positive. A larger coefficient indicates a larger effect of unexpected shocks on the conditional variance. While the standard GARCH model presents volatility as an additive function of lagged squared returns, the EGARCH model expresses volatility as an explicit multiplicative function of lagged innovations, which leads to a more complex functional dependence on innovations.[8] $\sigma_t^2 > 0$ is always fulfilled by definition without explicit non-negativity constraints due to its logarithmic formulation. The real-valued parameter ω represents the mean of the conditional volatility process as $\omega = E(\ln \sigma_t^2)$. In fact, equation (14) defines the (short memory variant of the)

[8] Demonstrated by several studies (e.g. Engle and Ng, 1993), a disadvantage of the EGARCH model is that it tends to overweight the impact of large shocks on volatility.

broader EGARCH family, which governs recently introduced exponential volatility models as special cases. The EGARCH can be expressed by a log-linear volatility process in terms of an infinite sum as:

$$\ln \sigma_t^2 = \omega + \sum_{k=0}^{\infty} \beta_k g_E(\varepsilon_{t-k-1}) \tag{14}$$

where ω is a real-valued constant term defined as $\frac{\alpha_0}{\beta(1)}$, β_k are real coefficients such that $\sum_{k=0}^{\infty} \beta_k^2 < \infty$ and g_E is a suitable, measurable function of ε_t. In the case of $\sum_{k=0}^{\infty} |\beta_k| < \infty$, long memory and integrated classes are omitted. If all roots of $\phi(z)$ lie outside the unit circle, the $\sum_{k=0}^{\infty} |\beta_k| < \infty$ condition is fulfilled and therefore $C_\beta = \sum_{k=0}^{\infty} \beta_k$ is well-defined. According to Theorem 2.1 in Nelson (1991), the log-volatility $\ln(\sigma_t^2)$ is strictly and weakly stationary and ergodic under the specified conditions. Let $\lambda = E(\ln \varepsilon_t^2)$ and $\eta_t = \ln \varepsilon_t^2 - E(\ln \varepsilon_t^2)$, then:

$$\ln r_t^2 = \omega + \lambda + \sum_{k=0}^{\infty} \beta_k g_E(\varepsilon_{t-k-1}) + \eta_t \tag{15}$$

which represents a linear signal plus noise process (Zafaroni, 2009). Furthermore, σ_t^2 and r_t^2 are also strictly stationary and ergodic with non-linear solutions:

$$\sigma_t^2 = e^{\omega} C_\varepsilon \prod_{k=0}^{\infty} \left(e^{\kappa \varepsilon_{t-k-1}} * e^{\gamma |\varepsilon_{t-k-1}|} \right)^{\beta_k} \tag{16}$$

$$r_t^2 = e^{\omega} C_\varepsilon \varepsilon_t^2 \prod_{k=0}^{\infty} \left(e^{\kappa \varepsilon_{t-k-1}} * e^{\gamma |\varepsilon_{t-k-1}|} \right)^{\beta_k} \tag{17}$$

where $C_\varepsilon = \{\exp[E(|\varepsilon_t|)]\}^{-\gamma C_\beta}$. Accordingly, the stationary solution of the return series r_t is $r_t = sign(\varepsilon_t)\sqrt{r_t^2}$.

Bollerslev and Mikkelsen (1996) developed the fractionally integrated EGARCH model (FIEGARCH), which models the logarithm of the conditional variance as a fractionally integrated process. A FIEGARCH(p, d, q) process can be expressed as follows:

$$\ln(\sigma_t^2) = \alpha_0 + \phi^{-1}(B)(1-B)^{-d} \psi(B) g_E(\varepsilon_{t-1}) \tag{18}$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_E(\varepsilon_{t-k-1}) \tag{19}$$

where $g_E(\varepsilon_t)$ is defined as above, $0 < d < 0.5$ is the long memory parameter that ensures covariance stationarity (Hosking, 1981), $\phi(z)$ and $\psi(z)$ are of orders p and q, respectively. Now, $\sum_{k=0}^{\infty} |\beta_k| = \infty$ and $\sum_{k=0}^{\infty} \beta_k^2 < \infty$.[9] The EGARCH is a special case of the FIEGARCH for $d = 0$. For $d = 1$ an

[9] Bollerslev and Mikkelsen (1996) suggest that allowing for $0.5 < d < 1$ can enhance the forecasting of shock effects in financial markets. Under this specification, $\ln \sigma_t^2$ remains strongly stationary but is no longer weakly stationary, as indicated by the divergence $\sum_{k=0}^{\infty} \beta_k^2 = \infty$. Consequently, the theoretical properties of such volatility models cannot be derived from the results established by Surgailis and Viano (2002).

IEGARCH model emerges. In accordance with Theorem 2.1 of Nelson (1991) $\{\ln \sigma_t^2\}$ is strictly stationary and ergodic for $d < 0.5$. The strictly stationary solution of σ_t^2 is non-linear and given by:

$$\sigma_t^2 = e^{\omega} \prod_{k=0}^{\infty} \exp[\beta_k g_E(\varepsilon_{t-k-1})] \tag{20}$$

Similarly, the strictly stationary solutions of r_t^2 and r_t can be obtained but are omitted to save space. Accordingly, Theorem 2.1 of Nelson (1991) holds, if κ and γ both do not equal zero. As indicated by the stationary results, the existence of finite moments of r_t^2 and σ_t^2 requires the existence of the moment generating function of ε_t. As specified by Theorem 2.2 in Nelson (1991), r_t^2 and σ_t^2 have finite moments of arbitrary order, if ε_t are generalized error distributed errors with tail-thickness parameter $v > 1$, which includes the conditional normal case (with $v = 2$).[10] The model has either no finite moments or all moments are finite. While the autocovariance structure higher-order moments have been extensively studied in the literature (e.g., Breidt et al., 1998; Surgailis and Viano, 2002; He et al., 2002), the invertibility of this model still remains an open question in the literature.[11]

2.5 (Long memory) Log-GARCH

The Log-GARCH model goes back to Geweke (1986), Pantula (1986) and Milhøj (1987) and exhibits a few advantages in comparison to the aforementioned EGARCH. Similar to the EGARCH model, the Log-GARCH models the logarithm of the conditional variance and is therefore log-linear, can be represented by well-known AR or ARMA processes, and guarantees positive conditional variance without the need for constraints on the parameters. As identified by Nelson (1991), and further corroborated by Francq et al. (2017) and Feng et al. (2020), the Log-GARCH contributes to the class of EGARCH models by serving as a special case with g_E being replaced by $g_{Log}(\varepsilon_t) = \ln \varepsilon_t^2 - E(\ln \varepsilon_t^2) = \eta_t$. Following Francq and Sucarrat (2013) and Sucarrat et al. (2016) the Log-GARCH is defined as:

$$\ln \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \ln r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \ln \sigma_{t-j}^2 \tag{21}$$

where α_0 controls the level of volatility in a multiplicative manner, α_i measures the impact of logarithmized past returns, and β_j measures the persistence in volatility. Let $\mu = E(\ln r_t^2)$, $\omega = E(\ln \sigma_t^2)$, $\tau = E(\ln \varepsilon_1^2)$, resulting in $\omega = \mu - \tau$, $\eta_t = \ln \varepsilon_t^2 - \tau$ and $\tilde{\varepsilon}_t^2 = e^{\eta_t}$, defining $\tilde{\varepsilon}_t = e^{-\tau/2}\varepsilon_t$, $\ln r_t^2$ has the following ARMA(p^*, q) representation, as noted by Francq and Sucarrat (2013; 2018):

$$\phi(B)(\ln r_t^2 - \mu) = \psi(B)\eta_t \tag{22}$$

[10] However, if the error process is i.i.d. ged with $v < 1$ or t-distributed, for any $k > 0$, $E(|r_t|^k) = \infty$ and $E(\sigma_t^{2k}) = \infty$ holds, which implies that the (FI-)EGARCH cannot be covariance stationary under t-distributed innovations.

[11] As initially noted by Granger and Andersen (1978), establishing the invertibility of nonlinear moving average models presents significant theoretical challenges.

where $\phi(B) = 1 - \sum_{i=1}^{p^*} \phi_i B^i = 1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j$ and $\psi(B) = 1 + \sum_{j=1}^{q} \psi_j B^j = 1 - \sum_{j=1}^{q} \beta_j B^j$. Feng et al. (2020) show that the process can be defined by an ARMA process:

$$\ln \sigma_t^2 = \omega + (\phi^{-1}(B)\psi(B) - 1)\eta_t \tag{23}$$

$$= \omega + \sum_{k=1}^{\infty} \beta_k \eta_{t-k} \tag{24}$$

When $\phi^{-1}(z)$ and $\psi(z)$ do not have common factors and their roots lie outside the unit circle, the aforementioned conditions in the EGARCH model description are satisfied. The invertibility of the Log-GARCH is discussed e.g. in Francq and Sucarrat (2018) and Francq and Zakoian (2019). To ensure the standard ARMA representation, it is also assumed that $E[(\ln \varepsilon_t^2)^2] < \infty$ (as stated in Example 2.2 of Franqc et al., 2013). A sufficient condition for this assumption is given by A1 in Feng et al. (2020). Assuming that these conditions hold, the conditional volatility process $\{\ln \sigma_t^2\}$ is strictly stationary and ergodic with finite moments of arbitrary order. The strictly stationary solution for the return process $\{r_t\}$ is nonlinear and of the following form:

$$r_t = \varepsilon_t e^{\frac{\omega}{2}} \prod_{k=1}^{\infty} (|\tilde{\varepsilon}_t|)^{\frac{\beta_k}{2}} \tag{25}$$

where $\tilde{\varepsilon}_t = \exp(\eta_t)$. Similarly, strictly stationary solutions of r_t^2 and σ_t^2 can be derived and are both products of powers of $|\varepsilon_t|$. However, these are omitted to save space. $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j$ indicates the persistence of volatility.[12] The Log-GARCH model exhibits some useful properties (Sucarrat, 2019): The volatility is bounded at 0 from below and allows negative autocorrelations of r_t^2, which is useful for modeling daily financial returns. A significant limitation of the Log-GARCH model was highlighted by Engle and Bollerslev (1986) concerning the model's non-existence in the presence of zero observations. Several potential solutions to this issue have been proposed, including the use of de-meaned returns (Harvey et al., 1994), a log-modulus transformation of the innovations (Feng et al., 2023), or replacing zero returns with optimal estimates (Sucarrat and Escribano, 2018).

The FI-Log-GARCH model was proposed by Feng et al. (2020) by adopting the ARMA representation and inserting the fractional difference operator. The FI-Log-GARCH is defined by:

$$\phi(B)(1 - B)^d(\ln r_t^2 - \mu) = \psi(B)\eta_t \tag{26}$$

One can see that the defined process follows a FARIMA representation (Hosking, 1981; Granger and Joyeux, 1980) of $\ln r_t^2$, which is well-defined under the conditions that $0 < d < 0.5$ (due to convergence and stationarity reasons), the innovation distribution satisfies some regularity conditions, and $\phi(z)$ as well as $\psi(z)$ do not have any common factors with all roots lying outside the unit circle. Additionally, it is required that $Pr(\varepsilon_t = 0) = 0$. The log-volatility series corresponds to the exponential

[12] For a Log-GARCH(1, 1), Francq et al. (2013) show that $|\alpha_1 + \beta_1| < 1$ is a sufficient condition for a stationary solution and ergodicity.

transformation of the conditional mean series in a fitted FARIMA model. If $d = 0$, the Log-GARCH results and if $q = 0$, a FAR model of $\ln r_t^2$ which can be represented as a FI-ARCH model (see e.g. Andersen et al., 2003; 2006) can be obtained. As $\ln(r_t^2)^m = m\ln(r_t^2)$, the FI-Log-GARCH model is power-transformation invariant (Feng et al., 2020). Another representation can be obtained by the ARMA representation:

$$\ln\sigma_t^2 = \omega + [(1-B)^{-d}\phi^{-1}(B)\psi(B) - 1]\eta_t \tag{27}$$

$$= \omega + \sum_{k=1}^{\infty}\beta_k\eta_{t-k} \tag{28}$$

where $\eta_t = \ln\varepsilon_t^2 - E(\ln\varepsilon_t^2)$ is an i.i.d. sequence with zero-mean and finite variance. If we define $\omega = E(\ln\sigma_t^2)$, $\tau = E(\ln\varepsilon_t^2)$ and $\tilde{\varepsilon}_t^2 = e^{\eta_t}$, $\tilde{\varepsilon}_t = e^{-\tau/2}\varepsilon_t$, we obtain the strictly stationary solution

$$\sigma_t^2 = e^{\omega}\prod_{k=1}^{\infty}|\tilde{\varepsilon}_{t-k}|^{\beta_k} \tag{29}$$

Defining $\beta_0 = 1$, $\beta_{max} = \max(\beta_k)$ and $\beta_{min} = \min(\beta_k)$ and assuming $E(\varepsilon_1^K) < \infty$, $E(r_1^4) < \infty$ if $K \geq 4\beta_{max}$ and $\beta_{min} > -1/4$. As the $\ln r_t^2$ of the FI-Log-GARCH is a linear FARIMA process (Feng et al., 2020), the $MA(\infty)$ (Beran et al., 2013; Pipiras and Taqqu, 2017) strongly and weakly stationary solutions for $\ln(r_t^2)$ are

$$\ln r_t^2 = \mu + \sum_{i=0}^{\infty}a_i\eta_{t-i} \tag{30}$$

where a_i are the coefficients of $\phi^{-1}(B)(1-B)^{-d}\psi(B)$. If $d = 0$, the coefficients decay at an exponential rate ($\sum_{i=0}^{\infty}|a_i| < \infty$) whereas $0 < d < 0.5$ leads to $\sum_{i=0}^{\infty}a_i = \infty$ but $\sum_{i=0}^{\infty}a_i^2 < \infty$. By replacing η_{t-i} with $g(\varepsilon_t) = \kappa\varepsilon_t + \gamma(|\varepsilon_t| - E(|\varepsilon_t|))$, we receive the FIEGARCH (Nelson, 1991; Bollerslev and Mikkelsen, 1996). Additionally, the FI-Log-GARCH is a limit FIAPARCH as $\delta \to 0$ with $\gamma_i = 0$ and a FIEGARCH with $g(\varepsilon_t) = \ln\varepsilon_t^2 - E(\ln\varepsilon_t^2)$. The parameters of the FI-Log-GARCH model can be estimated consistently if a positive higher-order moment of r_t^2 exists. The existence of higher-order moments is examined in Theorems 1 and 2 in Feng et al. (2020).

While in the Log-GARCH the volatility series and the squared return series are assumed to be log-linear, in the EGARCH only the conditional volatility series is log-linear. Feng et al. (2020) showed that r_t is weakly stationary and can have finite moments until any given order under $\beta_k > 0$ and finite moments of ε_1 of corresponding order. Therefore the Log-GARCH contains moments of any arbitrary order under generally weak and explicit conditions and hence has other conditions for the existence of finite moments than the EGARCH. While the EGARCH requires the existence of the moment generating function of the innovations, the Log-GARCH depends on the innovation distribution of some required order. However, the coefficients are now again subject to some restrictions[13], which is not the case in the EGARCH definition. Additionally, there is no problem with invertibility. However, the basic Log-GARCH does not account for the leverage effect, in the MA(∞) form negative coefficients are only allowed to some extent and influence the existence of high-order moments[14], and returns near zero may cause mathematical inefficiencies. Accordingly, recent studies (Francq et al., 2013; 2016; Feng et al., 2020) adjusted the Log-GARCH by introducing an asymmetric effect and long memory.

3 Novel Modified Members of the EGARCH-family

3.1 (Long memory) Modulus asymmetric Log-GARCH

To deal with the influence of returns close to zero, Feng et al. (2023) most recently introduced a log-linear (long memory) volatility model, which is called the modulus asymmetric (FI-)Log-GARCH ((FI-)MLog-GARCH). They use the modulus-log transformation $\zeta_t = sign(r_t)\ln(|\varepsilon_t| + 1)$ proposed by John and Draper (1980) with the power parameter set to zero (see Box and Cox (1964) for shifted log-transformations with power zero) and centered absolute log-transformations $[|\zeta_t| - E(|\zeta_t|)]$ for the asymmetric and magnitude terms. By assuming that the innovations ε_t follow a symmetric distribution process, $E(\zeta_t) = 0$ is satisfied by definition. Their proposal can be generalized by a new class of log-modulus volatility models called MLog-GARCH, defined by:

$$\ln \sigma_t^2 = \omega + \phi^{-1}(B)\psi(B)g_{MLog}(\varepsilon_{t-1}) \tag{31}$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_{MLog}(\varepsilon_{t-k-1}) \tag{32}$$

where $g_{MLog}(\varepsilon_t) = \kappa(\zeta_t(\varepsilon_t) - E(\zeta_t(\varepsilon_t))) + \gamma(|\zeta_t(\varepsilon_t)| - E(|\zeta_t(\varepsilon_t)|)$ and all the other terms are similarly defined as in 2.3. Due to the zero-mean construction of ζ_t, $\omega = E(\ln \sigma_t^2)$ holds. In addition to using the log-modulus transformation, $g_{MLog}(\varepsilon_t)$ is adapted directly from the EGARCH model, rather than from the Log-GARCH framework. As a result, $g_{MLog}(\varepsilon_t)$ is expressed as a function of the innovations, rather than observed returns. Therefore, this model presents another exponential volatility model with an asymmetric effect. Feng et al. (2023) also notice the different limit behavior of $ln(1 + |x|)$ when x approaches zero or infinity. Specifically, $ln(1 + |x|)$ approaches $|x|$, as $x \to 0$ but behaves like $ln\,|x|$, as $x \to \pm\infty$. Thus, the tail behavior is comparable to that of a Log-GARCH-type model. Now both sides of the volatility equation are expressed on a logarithmic scale. Therefore the prerequisites for the existence of finite fourth moments in this model are comparable to those of the Log-GARCH. The required conditions for the existence of finite moments are much weaker compared

[13] If $k \leq -1$, $E(\varepsilon_t^k) = \infty$ holds for any reasonable innovation distribution, which creates the restriction on the negative coefficients, assuming that the innovation density is bounded from zero.
[14] Therefore, some authors propose the non-negativity restriction to $\phi_i, \psi_j \geq 0$.

to the EGARCH and the process can exhibit finite moments of any given order, since for any $m > 0$, $E\{exp[g_{MLog}(\varepsilon_{t-k-1})]\}^m < \infty$, if $E(|\varepsilon_1|^m) < \infty$. Additionally, the constraint on negative coefficients in a Log-GARCH model is now removed, because $0 < (|\varepsilon_t| + 1)^m < 1$, for any $m < 0$. Near-zero innovations are handled in a nearly linear manner, effectively addressing the issue of zero or near-zero observations, because $\zeta_t = 0$ if $\varepsilon_t = 0$, is well-defined. As a result, the proposed model benefits from the strengths of both Log-GARCH and EGARCH. Assuming that κ and γ do not both equal zero, $g_{MLog}(\varepsilon_t)$ is an i.i.d. zero-mean sequence with $0 < var[g_{MLog}(\varepsilon_t)] < \infty$, so Theorem 2.1 in Nelson (1991) holds for the MLog-GARCH model which implies that $\ln \sigma_t^2$ is strictly stationary and ergodic. However, $exp\left[g_{MLog}(\varepsilon_{t-k-1})\right]$ is now of a power instead of an exponential form. Let C_ζ be defined in terms of C_β introduced in the EGARCH

$$C_\zeta = \{exp[E(|\zeta_t|)]\}^{-\gamma C_\beta}$$

Then the nonlinear strict stationary solutions of σ_t^2 and r_t are given by

$$\sigma_t^2 = e^\omega C_\zeta \prod_{k=0}^{\infty}\left[(|\varepsilon_{t-k-1}| + 1)^{sign(\varepsilon_t)\kappa} * (|\varepsilon_{t-k-1}| + 1)^\gamma\right]^{\beta_k} \tag{33}$$

$$r_t = e^{\omega/2}\sqrt{C_\zeta}\varepsilon_t \prod_{k=0}^{\infty}\left[(|\varepsilon_{t-k-1}| + 1)^{sign(\varepsilon_t)\kappa} * (|\varepsilon_{t-k-1}| + 1)^\gamma\right]^{\beta_k/2} \tag{34}$$

The absolute conversion of $ln[E(r_t^2)]$ documented in Feng et al. (2023) implies the absolute convergence of the infinite product of $E(r_t^2)$. For further details on these issues, we refer to Feng et al. (2023), Theorem 2.2.4 of Little et al. (2022), and formulas 0.252 and 0.255 from Gradshteyn and Ryzhik (2007). The condition of square summability of β_k is also sufficient for the convergence of $E(r_1^2)$ or equivalently of $var(r_1)$, as long as the necessary higher-order moments of ε_1 exist. Furthermore, it is necessary that $m \geq \beta_{max} = max[2, (\gamma - \kappa) * \beta_k, (\gamma + \kappa) * \beta_k]$, since $var(r_1)$ is not defined if $E(|\varepsilon_1|^m) = \infty$. Additionally, if $E(|\varepsilon_1|^m) < \infty$ for some $m \geq 2\beta_{max}, m \geq 3\beta_{max}, \ldots$, then the fourth, sixth, and higher moments of r_t are well defined and calculatable. Hence, the MLog-GARCH can possess finite moments up to any desired order, provided the innovation distribution has finite moments of the required order, similar to the Log-GARCH, which is satisfied for example by t-distributed innovations when the degrees of freedom are of required order. This beneficial property is not shared by models like the EGARCH, giving the proposed model a theoretical advantage. The MLog-GARCH is defined vastly different from the symmetric Log-GARCH but also to the asymmetric Log-GARCH defined in Franqc et al. (2013): It avoids the issue with zero observations and allows for all negative coefficients.

The long memory version of the modulus asymmetric Log-GARCH (FI-MLog-GARCH) was recently introduced by Feng et al. (2023) and is defined as follows:

$$\ln \sigma_t^2 = \omega + (1 - B)^{-d}\, \phi^{-1}(B)\psi(B)g_{MLog}(\eta_{t-1}) \tag{35}$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_{MLog}(\eta_{t-k-1}) \quad (36)$$

where $g_{MLog}(\varepsilon_t)$ is defined as above, $0 \leq d < 0.5$ is the long memory parameter and all other terms remain as in the FIEGARCH model. With the aforementioned statistical properties retaining validity, it is important to note that β_{max} is now jointly influenced by both short memory and long memory components, and is well-defined for all $0 \leq d < 0.5$, even though strong long memory may significantly impact its value. Similar to its short memory pendant, $g_M(\varepsilon_t)$ is a series dependent on the innovations rather than the observations. Under regularity conditions defined in Feng et al. (2023), the FI-MLog-GARCH can obtain finite moments up to any desired order, assuming that the innovation distribution has finite moments of the required order, which is a property shared with the FI-Log-GARCH. Nevertheless, similar to the (FI)EGARCH model, the invertibility of $\ln \sigma_t^2$ in the (FI-)MLog-GARCH is an open question in the literature and has not yet been established, as it is not as obvious as in the Log-GARCH and can only be proven by the convergence of a very complex recursive expansion of its definition in terms of r_t (Feng et al., forthcoming). It appears that proving this would require the use of a highly complex infinite product of infinite products, which is beyond the scope of this paper.

3.2 Double Power Modulus EGARCH and (Long memory) Modified EGARCH

Generalization of the previously discussed exponential volatility models leads to the so-called EGARCH family (Feng et al., forthcoming). Adjusting the general EGARCH framework by allowing unique and distinct power parameters for the sign and magnitude terms results in a double-power modulus EGARCH class, which might be an attractive alternative to the aforementioned volatility models:

$$\ln \sigma_t^2 = \omega + \phi^{-1}(B)\psi(B)g_{DP}(\varepsilon_{t-1}) \quad (37)$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_{DP}(\varepsilon_{t-k-1}) \quad (38)$$

with $g_{DP}(\varepsilon_t)$ being a suitable, measurable function in ε_t, such that $E[g_{DP}(\varepsilon_t)] = 0$ and $0 < var[g_{DP}(\varepsilon_t)] = E[g_{DP}^2(\varepsilon_t)] < \infty$. Therefore, $\{g_{DP}(\varepsilon_t)\}$ forms an i.i.d. zero-mean sequence with finite variance and $\ln \sigma_t^2$ is a stationary process. Consequently, $\ln r_t^2$ is a linear signal plus noise model as defined in Zaffaroni (2009) (Feng et al., forthcoming). The EGARCH family governs several models, which strongly depend on the type of choice of g_{DP}. These include the MEGARCH and FI-MLog-GARCH (Feng et al. 2023). Although the EGARCH family encompasses a broad class of models, it nevertheless constitutes a specific subclass within the broader category of exponential volatility models as defined in Zaffaroni (2009) and Surgailis and Viano (2002). Generalizing the EGARCH model, let:

$$g_{DP}(\varepsilon_t) = \kappa\{g_a(\varepsilon_t) - E[g_a(\varepsilon_t)]\} + \gamma\{g_m(\varepsilon_t) - E[g_m(\varepsilon_t)]\} \quad (39)$$

where κ and γ are two real-valued coefficients. It is easy to see that this generalized framework contains the EGARCH (Nelson, 1991) with $g_a(\varepsilon_t) = \varepsilon_t$ and $g_m(\varepsilon_t) = |\varepsilon_t|$, which consequently results in $E[g_a(\varepsilon_t)] = E[\varepsilon_t] = 0$ and $E[g_m(\varepsilon_t)] = E[|\varepsilon_t|]$. However, $g_a(\varepsilon_t)$ and $g_m(\varepsilon_t)$ can be any suitable transformation of ε_t. Generally, we propose the following two cases:

$$g_{1,a}(\varepsilon_t) = sign(\varepsilon_t)|\varepsilon_t|^{p_a}/p_a$$
$$g_{2,a}(\varepsilon_t) = sign(\varepsilon_t)[(|\varepsilon_t| + 1)^{p_a} - 1]/p_a$$

as well as (40)

$$g_{1,m}(\varepsilon_t) = |\varepsilon_t|^{p_m}/p_m$$
$$g_{2,m}(\varepsilon_t) = [(|\varepsilon_t| + 1)^{p_m} - 1]/p_m$$

The error transformation $g_{1,\cdot}$ performs a power transformation and $g_{2,\cdot}$ enacts a modulus as well as a power transformation simultaneously. As a special case, for $p_i = 0$, where $i \in \{a, b\}$, a log-transformation is performed and the division by p_i is dropped. Thereby, for example $g_{1,a}(\varepsilon_t) = sign(\varepsilon_t)\ln(|\varepsilon_t|)$ and $g_{2,a}(\varepsilon_t) = sign(\varepsilon_t)\ln(|\varepsilon_t| + 1)$ are employed for $p_a = 0$. It can be easily shown that the Log-GARCH can also be represented by setting $\kappa = 0$ and using $g_{1,m}(\varepsilon_t)$ setting $p_m = 0$.

Although the use of a single power parameter may lead to a clear limitation, according to model selection criteria the improvements caused by making use of two power parameters are generally insignificant (Feng et al., forthcoming). Based on empirical evidence, we propose to fix the two power parameters at $p_a = 0$ for the sign effect and $p_m = 1$ for the magnitude term, which leads to a novel exponential volatility model, where the log of the conditional variance is determined by combining equations of the traditional EGARCH defined in Nelson (1991) and the MLog-GARCH (defined above). Defining $p_m = 1$ results in keeping the power of the absolute magnitude term of the EGARCH (magnitude term of $g_E(\varepsilon_t)$) unchanged, while replacing its power of the sign effect with $p_s = 0$ ($g_{MLog}(\varepsilon_t)$ part for the modulus-log leverage term) culminates in the modified EGARCH (MEGARCH), defined by:

$$\ln \sigma_t^2 = \omega + \phi^{-1}(B)\psi(B)g_{ME}(\varepsilon_{t-1}) \quad (41)$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_{ME}(\varepsilon_{t-k-1}) \quad (42)$$

where $g_{ME}(\varepsilon_t) = \kappa\,[\zeta_t(\varepsilon_t) - E(\zeta_t(\varepsilon_t))] + \gamma[(|\varepsilon_t|) - E(|\varepsilon_t|)]$, while all other terms remain the same. Now the conditions for the existence of finite moments are analogous to those of the EGARCH model, as the tail behavior of $g_{ME}(\varepsilon_t)$ is primarily governed by the magnitude component. Testing a variety of different return series of different markets, this specific double power modulus EGARCH model proves to perform comparably well. The modulus transformation is well-defined over all real powers (including negative powers), which makes the approach very flexible. Feng et al. (forthcoming) note that the general performance of this volatility model class is improved, if non-integer $p_s < 1$ are allowed. The use and interpretation of negative power indices is still an open question in the literature and should be studied elsewhere. As $g_{ME}(\varepsilon_t)$ is a combination of $g_{MLog}(\varepsilon_t)$ and $g_E(\varepsilon_t)$, the stationary solutions of σ_t^2, r_t^2 and r_t are combinations of those of the traditional EGARCH and the MLog-GARCH:

$$\sigma_t^2 = e^{\omega} C_\varepsilon \prod_{k=0}^{\infty} \left[(|\varepsilon_{t-k-1}| + 1)^{sign(\varepsilon_t)\kappa} * e^{\gamma|\varepsilon_{t-k-1}|}\right]^{\beta_k} \tag{43}$$

$$r_t = e^{\omega/2} \sqrt{C_\varepsilon} \varepsilon_t \prod_{k=0}^{\infty} \left[(|\varepsilon_{t-k-1}| + 1)^{sign(\varepsilon_t)\kappa} * e^{\gamma|\varepsilon_{t-k-1}|}\right]^{\beta_k/2} \tag{44}$$

where C_ε is similarly defined as in the EGARCH model description depicted above. It should be noted that the centralization of the magnitude term in the short memory case is not needed because $\ln \sigma_t^2$ can be defined equivalently with $(|\varepsilon_{t-k-1}| - E[|\varepsilon_{t-k-1}|])$ being replaced by $|\varepsilon_{t-k-1}|$ and a correspondingly adjusted intercept ω^*. An alternative representation of the MEGARCH yields a closed-form ARMA representation, which is more practical for use. For simplicity, consider the MEGARCH(1, 1), which can equivalently be expressed as:

$$\ln \sigma_t^2 = \alpha + \beta \ln h_{t-1} + \{\kappa\,[\zeta_t - E(\zeta_t)] + \gamma[(|\varepsilon_t|) - E(|\varepsilon_t|)]\} \tag{45}$$

where $|\beta| < 1$ and $\alpha = (1 - \beta)\omega$. Since this paper only considers model orders $(1, 1)$ the ARMA-representation will be used in the application part. However, the extension of this definition to a higher order MEGARCH(p, q) is straightforward.

Introducing the fractional differencing operator in the AR polynomial, the long memory extension of the proposed model is straightforward and called the FIMEGARCH(p, d, q):

$$\ln \sigma_t^2 = \omega + (1 - B)^{-d}\, \phi^{-1}(B)\psi(B) g_{ME}(\varepsilon_{t-1}) \tag{46}$$

$$= \omega + \sum_{k=0}^{\infty} \beta_k g_{ME}(\varepsilon_{t-k-1}) \tag{47}$$

with all other components remaining the same. This novel exponential long memory volatility model is a combination of the FIEGARCH introduced by Bollerslev and Mikkelsen (1996) depicted above and the FI-MLog-GARCH defined in Feng et al. (2023). As $g_{ME}(\varepsilon_t)$ is defined as in the short memory case, the stationary solutions of the FIMEGARCH are indeed again combinations of the FIEGARCH and the FI-MLog-GARCH. Under regularity conditions following Theorem 2.1 of Nelson (1991) and assuming that $d < 0.5$, the volatility series $\{\ln \sigma_t^2\}$ is strictly stationary and ergodic and the strictly stationary solutions of σ_t^2 and r_t can be obtained as follows:

$$\sigma_t^2 = e^{\omega} \prod_{k=0}^{\infty} \exp[\beta_k g_{ME}(\varepsilon_{t-k-1})] \tag{48}$$

$$r_t = e^{\omega/2} \varepsilon_t \prod_{k=0}^{\infty} \exp[\beta_k g_{ME}(\varepsilon_{t-k-1})/2] \tag{49}$$

which are all of simple explicit power forms and the last formula provides a method to simulate return observations of a given FIMEGARCH. Because $1 \leq (|\varepsilon_t| + 1) \leq e^{|\varepsilon_t|}$, the conditions for the existence of finite moments are exactly the same as those for the (FI-)EGARCH, which implies that negative coefficients are allowed. Consequently, Theorem 2.1 of Nelson (1991) applies and the model has finite moments of arbitrary order under regulatory conditions. Similar to the FIEGARCH, the invertibility of $\ln\sigma_t^2$ or σ_t^2 is not established in the literature and is beyond the aim of this paper.

4 Maximum Likelihood Estimation

For practical implementation of the aforementioned short and long memory volatility models, a QMLE procedure is applied. While the parameters of the short memory volatility models can be obtained through exact finite-order representations, the long memory processes can only be estimated via their $AR(\infty)$ representation, with the coefficients truncated at a finite lag M (here $M = 1000$ is applied). It is assumed that the QMLE of members of the EGARCH familiy should be $\sqrt{n}$-consistent and asymptotically normally distributed (which is a common conjecture and therefore does not imply any restrictions). Even if those statistical properties are confirmed empirically, the general mathematical proof is still missing as the proof of the invertability of EGARCH models is still not established (Straumann and Mikosch, 2006). This analysis is devoted to the implementation of the estimation procedure for the proposed models under the assumption of conditionally normally distributed innovations for simplicity. Let θ represent a generic parameter vector. Given the assumption that the fourth moment exists, the quasi-conditional Gaussian maximum log-likelihood estimator for estimating the proposed models based on the observable time series r_t, $t = 1, \dots, n$ is defined as follows:

$$L(\theta) = \frac{1}{n}\sum_{t=1}^{n} l_t \ with \ l_t = -\frac{1}{2}\left\{\ln(2\pi) + \ln[\sigma_t^2(F_{t-1};\theta)] + \frac{r_t^2}{\sigma_t^2(F_{t-1};\theta)}\right\} \tag{50}$$

where σ_t is the conditional volatility estimated by the implemented GARCH model. Maximizing $L(\theta)$ the maximum likelihood estimates $\hat{\theta}$ represent asymptotically normally distributed, consistent, and efficient estimates of the parameters under certain regularity conditions (such as correct model specification, weak stationarity, and ergodicity). However, the procedure assumes a conditional normal distribution. If the volatility recursion or required regularity conditions are misspecified, QMLE consistency may fail; under appropriate conditions, Gaussian QMLE remains consistent despite non-Gaussian innovations. By using the quasi-maximum likelihood method, which is applied here, the assumption of a normal distribution can be maintained even if the true distribution is different because the parameter estimates are still asymptotically normal and consistent (Bollerslev and Wooldridge, 1992). The loss of efficiency is small and negligible considering that the method compensates for inconsistent estimates if the normal distribution assumption does not hold.

Because of the similarities in the volatility formulations, the QMLE algorithms in R for estimating the EGARCH family models are quite similar. The generic parameter vector for the (long memory) models contains estimates for $\theta = (\omega, \phi, \kappa, \gamma, d)'$. Estimating the exponential heteroskedasticity models via algorithms in R, nonlinear optimization is performed using the augmented Lagrange method, originally proposed by Ye (1997), through the R function 'solnp' implemented by Ghalanos and Theussl (2011). To obtain the covariance matrix of the estimates the Hessian of the augmented problem at the optimal solution is inverted. Hence, the standard errors, $se(\hat{\Theta})$, are obtained by taking the square root of the diagonal elements of the variance-covariance matrix. Extensive application of these algorithms to numerous examples demonstrates that they perform efficiently and stably. The runtime is also practically acceptable. The long memory parameter is estimated over $d \in (0, 0.5)$ with an initial value $d_{start} = 0.25$. For simplicity, only order $(1, d, 1)$ models are considered.

The selection of the most appropriate model is guided by the Bayesian Information Criterion (BIC)[15], which balances model fit and complexity by penalizing the number of estimated parameters, thereby reducing the risk of overfitting. While the BIC is consistent and rewards models with higher likelihoods, it imposes a stronger penalty for model complexity as the sample size increases.[16] Unlike the Akaike Information Criterion (AIC), which prioritizes predictive accuracy and may favor more complex models, the BIC is more conservative, making it particularly suitable for selecting parsimonious models in large samples. In practice, the model with the lowest BIC value is considered the most suitable among the candidates. In the context of volatility modeling or time series analysis, the BIC provides a rigorous, information-theoretic basis for choosing between competing model specifications. To measure the performance of the recently introduced volatility models, empirical results are contrasted with those from GARCH, APARCH, FIGARCH, and FIAPARCH benchmark models fitted using the R package 'fEGarch' (Schulz et al., forthcoming). It is noted that the estimation processes for GARCH and APARCH models in both 'rugarch' and 'fEGarch' can occasionally struggle with convergence. In contrast, this issue usually did not occur when fitting the models with the algorithms proposed in this paper to estimate the exponential volatility models. The estimation procedure consistently performed stably. This indicates that using QMLE to estimate these models is a sound and effective approach.

In theory, applying the QMLE requires that σ_t^2 is invertible, meaning it should be expressible as a formula for r_t that converges almost surely (Straumann and Mikosch, 2006). Yet, establishing the invertibility of the standard EGARCH model remains an unresolved issue in the literature. Some results related to the invertibility of the EGARCH(1,1) model can be found in Wintenberger (2013). Similarly, the question of invertibility for the MLog-GARCH and MEGARCH models has yet to be addressed.

[15] The BIC is formally defined as:

$$BIC = -2 * \widehat{LL}/n + k * \ln(n)/n$$

where $\widehat{LL}$ is the maximized Log-Likelihood of the specific GARCH model, k is the number of estimated parameters, and n is the sample size (Schwarz, 1978; Ghalanos, 2017).

[16] One of the key statistical properties of the BIC is its consistency: as the sample size tends to infinity, the BIC will select the true model (assuming it is among the candidates) with probability approaching one (Hannan, 1980).

However, in the literature empirical evidence demonstrates that the QMLE of the EGARCH models works very well in practice (e.g. Perez and Zaffaroni, 2008).

Once the models have been correctly specified and the optimal model for each country has been selected, a comparative analysis is conducted across the estimated coefficients. This comparison enables the identification of country-specific characteristics, particularly regarding the degree of volatility persistence, magnitude, and persistence of shocks affecting short-term volatility. Moreover, the analysis provides insights into the existence and intensity of leverage effects. By contrasting mature financial markets with newly emerging market economies, the study highlights structural distinctions in volatility behavior, which may be driven by differing market dynamics, regulatory environments, and levels of financial development.

5 Enhanced comparative empirical study

The dataset comprises daily closing prices from twenty national stock market indices. To evaluate the performance of the proposed models across diverse market conditions and to ensure data heterogeneity, historical return series from major stock indices including both mature financial markets from G10 countries and emerging markets, incorporating BRIC nations, in Asia, Europe, and the USA are analyzed. The selected indices include: AEX Index (AEX), Austrian Trades Index (ATX), CAC 40 (CAC), S&P/TSX Composite (CAD), DAX 40 (DAX), Dow Jones Industrial Average (DJI), EURO STOXX 50 (EST), FTSE 100 (FTS), Irish Stock Exchange Index (ISQ), Hang Seng Index (HSI), Korea Stock Exchange Composite (KOR), Madrid Stock Exchange General Index (MAD), Mexico IPC Index (MEX), NASDAQ Composite (NSQ), Nikkei 225 (NIK), New York Stock Exchange Index (NYS), Portugal PSI General Index (PSI), Russell 2000 (RUS), Standard & Poor's 500 Index (S&P), and Swiss Market Index (SWI). This cross-regional dataset enables a comprehensive and internationally diversified assessment of model robustness, robust cross-market comparisons, and enhances the generalizability of the findings regarding model performance in different economic and financial environments. The study spans 25 years, from January 1, 2000, to December 31, 2024, which is sufficiently large, encompassing significant global financial events with possible long term volatility dynamics. The most relevant and influential, which can be considered as the three major benchmark crises of the 90s and the new millennium, are the Dot-com Bubble (2000–2002), the Subprime Mortgage Crisis (2007–2009), and the COVID-19 Pandemic (2020–2022). These events serve as critical benchmarks for analyzing market volatility, its persistence and resilience. Consequently, the application of long memory models is both theoretically justified and empirically consistent. All data are downloaded from Yahoo Finance using the ‘yfinance’ Python package.

The indices exhibit common stylized facts such as volatility clustering, serial correlation of squared returns, and leptokurtosis, though the intensity of these characteristics varies among them.[17] Pre-diagnostic tests are conducted to detect the presence of ARCH effects and serial correlation of squared returns. These tests include autocorrelation functions (ACF), the ARCH-LM test (Engle, 1982), and the Ljung-Box test (Ljung and Box, 1978).

As a descriptive example, which will be used to describe the methodological procedure visually, Figure 1 shows the price and return series of the Standard and Poor's 500 (S&P 500). As a free-float, market-capitalization-weighted index, the S&P 500 is one of the most widely followed equity indices globally, reflecting the performance of 500 leading publicly traded companies listed on U.S. stock exchanges. Due to its comprehensive nature, the S&P 500 serves not only as a barometer for the U.S. equity market, economic and corporate performance but also plays a key role in economic forecasting, including its incorporation into the Conference Board Leading Economic Index (Conference Board, 2025). From a historical and financial perspective, the S&P 500 exhibits significant volatility patterns that reflect underlying macroeconomic events. The index experienced a major peak in the late 1990s during the Dot-com Bubble, followed by a substantial decline coinciding with the 2000–2001 crash and the 9/11 terrorist attacks. A period of relative market stability followed, known as the "Great Moderation" (2002–2006), marked by low volatility fluctuations within a ±3% range (Reinhart and Rogoff, 2008). Disrupted by the 2007–2009 Global Financial Crisis, originating from the Subprime Mortgage Bubble, the index saw extreme price movements of up to ±10%. The post-crisis recovery was gradual, with volatility stabilizing at approximately ±5% until 2012. Volatility clustering is evident in the S&P 500 return series, aligning with Mandelbrot's (1963) observations in fractal market theory. Empirical analysis using autocorrelation functions and Ljung-Box tests reveals significant serial correlation, especially in the squared return series, suggesting persistent conditional heteroskedasticity. This is further supported by high ARCH effects, as demonstrated by significant ARCH-LM test statistics.

[17] Volatility clustering refers to periods of high volatility followed by high volatility and low volatility followed by low volatility, a phenomenon first noted by Mandelbrot (1963) and further analyzed by Ding et al. (1993). Serial correlation indicates that returns are not independent over time.

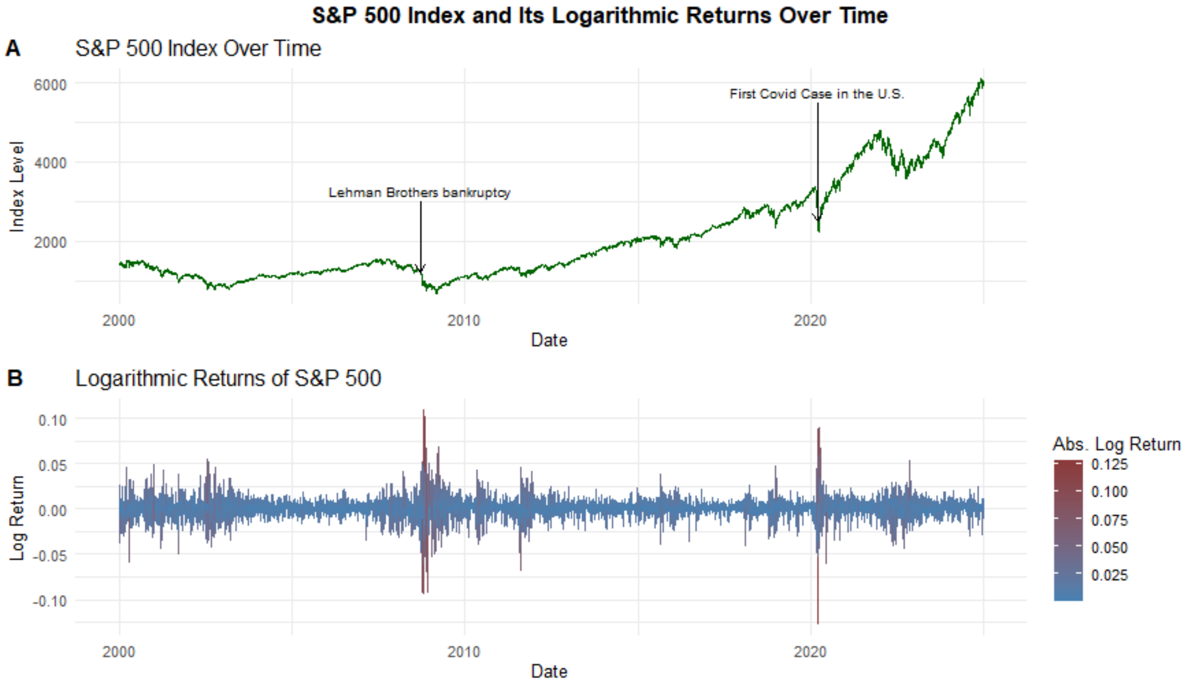

Figure 1 -Daily Price and Return Series of the S&P 500 over the sample period from January 2000 to December 2024

Panel A presents the time series of the S&P 500 index, showing its overall growth trajectory punctuated by major economic downturns and crises. Panel B displays the logarithmic returns of the S&P 500, emphasizing the fluctuations in daily price movements and their varying magnitudes, particularly in times of financial distress. The Lehman Brothers bankruptcy (2008 Global Financial Crisis), which was a major structural break in financial markets associated with heightened return volatility[18], and the first confirmed COVID-19 case in the U.S. (2020 Pandemic Shock), which constituted a significant exogenous shock that triggered an extreme return fluctuation and initially led to a steep market crash, followed by a rapid recovery due to monetary and fiscal stimulus interventions as well as a significant increase in retail trading are highlighted. These events correspond to sharp movements in both the index and the return series, reflecting crisis-induced market volatility spikes, highlighting the market's reaction to systemic risk, and illustrate the well-documented phenomenon of financial market cyclicality, where extended bull markets are periodically interrupted by crises and corrections (see Reinhart and Rogoff, 2008). These shocks led to a temporary breakdown of normal market functioning, leading to increased uncertainty and risk premia.

The logarithmic return series presents several key stylized facts well-documented in financial econometrics (Cont, 2001; Andersen et al., 2001). While there are alternating periods of low and high volatility, with volatility persisting over time, often described as volatility clustering, large returns tend to be followed by large returns (of either sign), confirming the presence of heteroskedasticity in financial time series. Additionally, extreme movements occur more frequently than would be expected under a

[18] The collapse of Lehman Brothers is considered a pivotal moment in the global financial crisis, prompting the United States to introduce the Troubled Asset Relief Program (TARP) as a measure to restore financial stability.

normal distribution, as indicated by sharp return spikes during crises. This suggests the presence of fat tails, a feature often modeled using GARCH-type processes (Bollerslev, 1986). While return magnitudes increase during crises, they eventually subside, returning to normal levels over time, aligning with empirical findings that market shocks have long memory but ultimately mean revert (Engle and Patton, 2001). The duration of volatility shocks is intrinsically linked to the interpretation of persistence and primarily involves the convergence behavior of the autocorrelation function toward its unconditional level (Engle and Bollerslev, 1986; French et al., 1987).[19] In cases where volatility shocks exhibit a high degree of persistence, the entire long-term structure of volatility may be affected, potentially resulting in a structural change in scale (see Feng, 2004; Feng and Sun, 2015).

Figure 2 addresses stylized facts regarding autocorrelation structures. In particular, as shown in Panel A, logarithmic returns are largely unpredictable and exhibit negligible autocorrelation. Knowing yesterday's S&P 500 return gives essentially no advantage in predicting today's return; the series behaves nearly like white noise. This aligns with extensive evidence that stock index returns have little to no linear predictability, reflecting the efficient market hypothesis in which any predictable patterns in returns are arbitraged away (Fama, 1970). This absence of serial correlation in log returns justifies treating the mean process as ARMA(0,0) (i.e. no ARMA structure).[20]

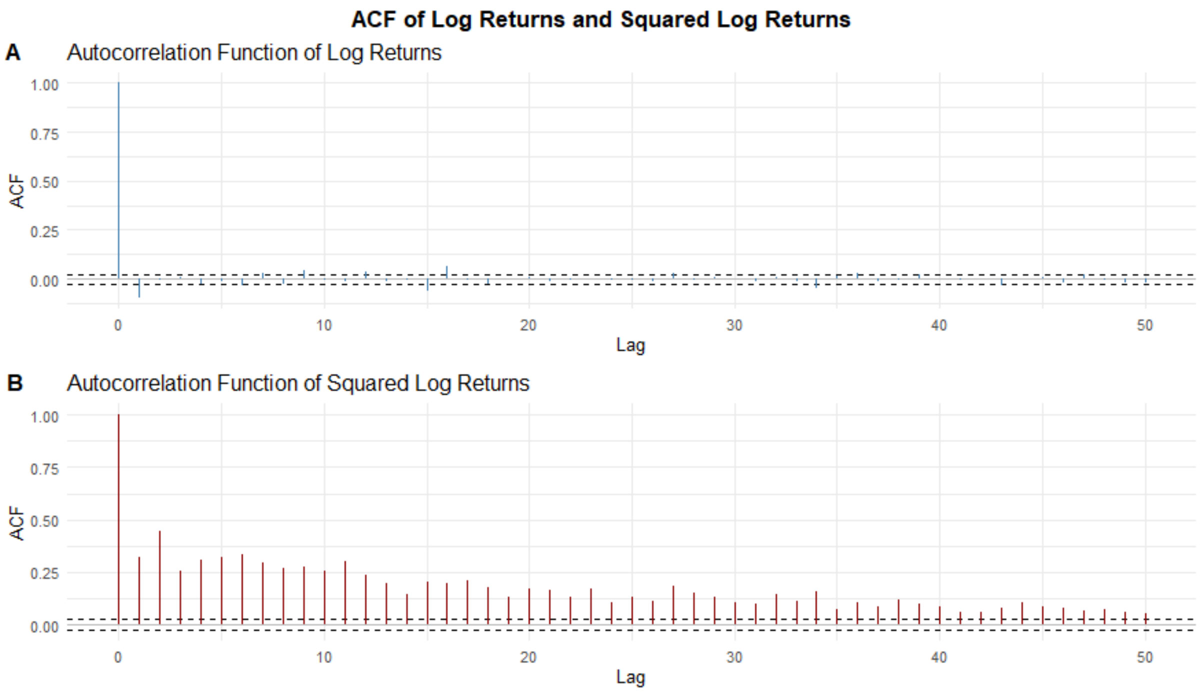

Figure 2 - Autocorrelation Function of Daily (Squared) Log-Returns of the S&P 500 up to Lag 50

[19] The interpretation of persistence acknowledges the existence of multiple conceptualizations in the academic literature (see Nelson, 1990). In the context of volatility, persistence refers to the extent to which shocks to the conditional variance influence future volatility forecasts, which is determined by the decay rate of the autocorrelation function of the variance process.

[20] It is worth noting that minor exceptions can occur at the very first lag due to microstructure effects or nonsynchronous trading, but even when a first-lag autocorrelation is detected, it tends to be small (on the order of a few percent or less) and quickly vanishes for higher lags.

By contrast, the volatility of returns tends to cluster in time: periods of high variability are followed by high variability, and vice versa. This phenomenon, first noted by Mandelbrot (1963), manifests as a slowly decaying autocorrelation function (ACF) for squared or absolute returns and thus a strongly persistent autocorrelation pattern (Figure 2 panel B) indicating time-varying conditional variance - a clear violation of the constant-variance assumption of homoscedastic models. Studies by Ding et al. (1993) document this "slowly decaying autocorrelation function" in absolute and squared returns across a variety of assets and observe that the decay can often be fitted more accurately by hyperbolic or long memory patterns rather than the exponential decay of short memory processes. The persistence in squared returns implies that shocks to volatility are long-lasting. The Ljung-Box (8748.6) as well as the ARCH-LM test statistic (1843.6) confirm this visual inspection. The economic intuition is that information arrivals are clustered in time. During turbulent periods, news (earnings shocks, macro surprises, etc.) come in flurries, keeping volatility high until the flow of new information abates. Thus, volatility tends to mean-revert slowly rather than instantly, creating observable dependence in second moments even as first-moment dependencies remain absent. Numerous papers have applied GARCH-family models as parsimonious tools to stock indexes, finding significant ARCH terms that corroborate the presence of autocorrelated variance (e.g., Engle and Bollerslev (1986) on equity index returns, Taylor (1986) on stock indices, Schwert (1989) on changing stock volatility). Unreported diagnostic test results for the remaining analyzed stock indices confirm the conjectures made about the S&P 500 for markets around the world. The ARCH-LM test results consistently show high test statistics for squared returns, strongly indicating the presence of ARCH effects in all indices. Additionally, the Ljung-Box statistics applied to squared returns are also notably large, implying significant serial correlation in the second moments of the return series. This persistence in volatility over time further particularly supports the application of long memory models from the GARCH family.

Overall, the statistical properties of the return series across all markets violate assumptions of constant variance. Therefore, improved model specification is expected through the application of models that incorporate weakened moment conditions as proposed by Ding et al. (1993), as well as those that explicitly account for asymmetric news impact structures, as suggested by Engle and Ng (1993). These modeling approaches enhance the ability to capture the empirically observed features of financial return series, such as heavy tails and asymmetric volatility responses to positive and negative shocks.

The previously described (historical) volatilities are subsequently estimated using univariate parametric volatility models. The availability of GARCH model implementations in R has significantly enhanced their accessibility and practical applicability in empirical research. The volatility models incorporating both short and long memory dynamics are denoted as GARCH (G), Fractionally Integrated GARCH (FIG), Asymmetric Power ARCH (APA), Fractional Integrated Asymmetric Power ARCH (FIAPA), Exponential GARCH (EG), Fractionally Integrated Exponential GARCH (FIEG), Modified Exponential GARCH (MEG), Fractionally Integrated Modified Exponential GARCH (FIMEG), Modulus-Adjusted Asymmetric Log-GARCH (MLG), and its fractionally integrated counterpart (FIMLG). Tables 1 and 2 present the corresponding log-likelihood values (LLik), the BIC, and, for models exhibiting long memory, the estimated long memory parameter $\hat{d}$ along with its associated standard error (se) in brackets. For each case, the model yielding the lowest BIC value is highlighted in bold.

According to the BIC, the FIEGARCH is in eight, the FIMEGARCH is in seven, the APARCH is in three, and the short memory variant of the modified EGARCH is in two of the twenty cases the best model. Interestingly, the modulus adjusted Log-GARCH (in its short and long memory variant) does not seem to be adequate enough to be selected as the best model in neither case.[21] The results indicate that the proposed modulus-adjusted exponential GARCH models (in their short and long memory variants) demonstrate strong practical applicability. A closer inspection of the differences in log-likelihood values reveals that, while some discrepancies are pronounced, others are relatively minor. Notably, the FIMEGARCH exhibits a comparable goodness-of-fit to the well-established FIEGARCH model in many instances. Across the evaluated cases, the long memory parameter estimated by the FIMEGARCH mostly adheres to expected ranges.

Across the twenty major equity indices, the models with more advanced features consistently provided better fit (lower BIC) than the basic GARCH(1,1). In particular, models that incorporate fractional integration (long memory) tended to outperform their short memory counterparts in nearly all markets. In contrast to Conrad et al. (2008), who often identified the FIAPARCH model as the best-fitting specification for volatility, the study at hand shows that the exponential heteroskedasticity models of the EGARCH family usually outperform these models. Interestingly, the predominant selection of the exponential models indicates that the gains in log-likelihood from capturing long-range dependence and asymmetry outweigh the penalty for added complexity. The estimation results strongly indicate that the best-performing model across all examined markets almost always incorporates the presence of long memory (high persistence) in stock return volatility, is always a variant that also includes the asymmetry feature and often embodies an important return transformation, confirming that capturing these complex features yields a significantly improved goodness-of-fit. The fractionally integrated models produce an estimate $\hat{d}$ that is significantly positive for most indices, typically in the range of about 0.3 to 0.5, indicating that volatility exhibits hyperbolic, slowly decaying autocorrelation. A surge in volatility due to a market shock tends to gradually mean-revert but at an exceedingly slow rate, leaving a "memory" in the conditional variance for many lags. The fractionally integrated models incorporating these features (like FIEGARCH and FIMEGARCH) are consequently frequently ranked at the top, whereas the classic GARCH and other non-integrated models provided only in a few cases the best in-sample fit, underscoring the importance of accounting for high persistence in volatility dynamics. The selection of

[21] However, also having tested the models on a large number of individual stock returns, the FI-MLog-GARCH seems to perform quite well on more volatile assets. This might serve as a preliminary indication for the modulus-asymmetric Log-GARCH to better handle more extreme innovations as the stock indices incorporated in this study entail numerous individual stocks from different branches and incorporate diversification effects implicitly.

the FIMEGARCH as the second most selected model indicates that alternative return transformations can sometimes capture distributional nuances better (Nugroho et al., 2021).

Crucially, the long memory parameters appear remarkably stable across regions. Both mature markets and emerging markets, despite their idiosyncrasies, exhibit fractional differencing estimates of similar order of magnitude. This suggests that long memory in volatility is a pervasive phenomenon not limited to a particular type of market. The persistence of volatility is universal in the sense that developed markets and emerging markets both require fractional dynamics to adequately capture their volatility clustering; even markets that are smaller or subject to more institutional frictions demonstrate protracted volatility decay, making risk modelling and forecasting challenging without long memory models. One possible explanation is that heterogeneous trading behaviors and information flows, which give rise to long memory, are present in all markets to some degree. This stability of the long memory parameter across diverse markets reinforces the idea that the intrinsic heterogeneity and clustering of volatility have a common structural character worldwide. In terms of volatility level and responsiveness, emerging markets are often more volatile on average and see larger shocks (due to lower liquidity or more turbulent economic conditions). Our findings suggest that this does not fundamentally change which model is appropriate. It is noteworthy that the estimated long memory parameters $\hat{d}$ in the FIMEGARCH and FI-MLog-GARCH models are generally lower than those obtained from the FIEGARCH and FIAPARCH models, which suggests that the inclusion of power or modulus transformations may contribute to a more stable and stationary estimation process. These transformations likely enhance the model's ability to capture the underlying volatility dynamics more efficiently, reducing the risk of overestimating persistence in the volatility structure.

In specific cases, the estimation of FIAPARCH - and in some instances even standard APARCH - models fails to converge or the computation of the inverse of the Hessian fails due to singularity.[22] Despite these estimation challenges, the parameters obtained from the successfully estimated alternative models demonstrate robustness and consistency across all examined stock indices, reinforcing the reliability of the overall empirical findings.

The estimated coefficients for all selected models, along with their standard errors (in parentheses), are presented in Tables 3 and 4. For convenience, we split the coefficient tables as the model constructions of the selected and optimal volatility processes differ significantly. Table 3 contains the parameter estimates of the EGARCH family, while Table 4 contains the estimates for the APARCH models. Results for additional cases are omitted for brevity. The short-term conditional volatility and associated risk characteristics of each country are reflected in the estimated coefficients of the best-fitting GARCH-type model. These coefficients determine whether short-term volatility primarily arises from new information - captured by the autoregressive terms - or from the persistence of its own past values -

[22] This lack of convergence may be attributed to limitations in sample size, which, although relatively large, may still be insufficient to support the complex parameter structure of these models.

represented by the conditional variance terms. To assess the presence and nature of asymmetric effects in volatility, in APARCH-type models, asymmetry is typically captured by the parameter $\hat{\gamma}$, which indicates whether past negative shocks have a statistically significant and disproportionate impact on current volatility. A positive and significant $\hat{\gamma}$ implies that negative innovations increase volatility more than positive ones of equal magnitude (Ding et al., 1993; Glosten et al., 1993). In contrast, the EGARCH-type models account for both the sign and size of past shocks through their logarithmic specification. Here, the asymmetry parameter determines the directional impact of past innovations on volatility, while the magnitude parameter captures the size effect. A negative $\hat{\kappa}$ indicates that negative shocks lead to greater increases in volatility than positive shocks. Furthermore, the non-linear structure of the EGARCH model ensures that large shocks, regardless of sign, tend to increase volatility more than smaller shocks, reflecting volatility clustering and the stylized fact of heavy-tailed return distributions.

The results indicate that, despite structural differences and varying estimated parameters, the FIMEG and FIEG models exhibit comparable empirical performance with parameter estimates of a similar order of magnitude. Consequently, the choice between the two models should be guided by the specific characteristics of the dataset under investigation.

Model	Statistic	S&P	DJI	NSQ	RUS	FTS	DAX	CAC	EST	NIK	HSI
G	LLik	20301.50	20629.03	18673.44	18372.64	20541.84	19136.28	19402.01	13435.08	17958.9	18118.83
	BIC	-6.450631	-6.554789	-5.932881	-5.837224	-6.502246	-6.024506	-6.06808	-6.029334	-5.858434	-5.878026
FIG	d	0.7048	0.7068	0.6997	0.5577	0.5879	0.6200	0.6657	0.5467	0.6211	0.6388
	(se)	0.0534	0.0567	0.0577	0.0596	0.0443	0.0465	0.0498	0.0454	0.0778	0.0761
	LLik	20280.65	20615.34	18641.81	18354.79	20538.02	19125.99	19368.52	13453.11	17939.24	18060.96
	BIC	-6.442609	-6.549044	-5.921434	-5.830157	-6.499649	-6.019884	-6.056226	-6.035547	-5.850594	-5.857815
APA	LLik	20451.70	20769.06	18775.47	18457.07	20698.31	19292.06	19584.45	13589.01	18040.13	18160.97
	BIC	-6.495616	**-6.596541**	-5.962548	-5.861290	-6.549044	**-6.070833**	-6.122449	-6.094724	-5.882112	-5.888874
FIAPA	d	0.4544	0.4909	0.5000	0.4664	0.4746	0.4897	0.5000	0.5000		
	(se)	0.0309	0.0400	0.0386	0.0426	0.0340	0.0357	0.0356	0.0403	f.t.c.	f.t.c.
	LLik	20447.47	20756.92	18697.16	18439.08	20685.07	19261.16	19515.43	13594.37		
	BIC	-6.492879	-6.591288	-5.936252	-5.854180	-6.543464	-6.059718	-6.099471	-6.095244		
EG	LLik	20425.89	20744.94	18752.43	18447.13	20681.01	19269.34	19566.42	13577.99	18035.93	18148.72
	BIC	-6.490188	-6.591651	-5.958003	-5.860913	-6.546336	-6.066433	-6.119548	-6.093546	-5.883587	-5.887733
FIEG	d	0.4017	0.4017	0.4763	0.4595	0.5000	0.4691	0.4578	0.4508	0.0000	0.4712
	(se)	0.0399	0.0399	0.0353	0.0417	0.0629	0.0336	0.0298	0.0385	0.0000	0.0520
	LLik	20457.06	20756.49	18793.94	18464.06	20697.02	19284.56	19584.16	13594.45	0.07398600	18161.95
	BIC	-6.498709	-6.593934	**-5.969812**	-5.864904	**-6.550022**	-6.069852	**-6.123728**	**-6.099056**	-5.882186	-5.890611
MEG	LLik	20437.86	20755.40	18753.94	18452.76	20685.46	19273.48	19566.39	13583.89	18039.08	18154.89
	BIC	-6.493995	-6.594976	-5.958482	-5.862703	-6.547746	-6.067739	-6.119537	6.096198	**-5.884616**	-5.889736
FIMEG	d	0.3390	0.3389	0.4402	0.4161	0.4917	0.4255	0.4132	0.4043	0.0000	0.4565
	(se)	0.0469	0.0469	0.0314	0.0462	0.0376	0.0405	0.0394	0.0441	0.0023	0.0576
	LLik	20461.60	20762.56	18792.21	18465.09	20694.48	19284.06	19579.70	13593.17	18039.16	18165.63
	BIC	**-6.500153**	-6.595865	-5.969262	**-5.865232**	-6.549216	-6.069693	-6.122333	-6.098479	-5.883221	**-5.891806**
MALG	LLik	20417.94	20732.49	18739.48	18435.07	20676.89	19262.38	19556.27	13579.49	18017.58	18137.37
	BIC	-6.487659	-6.587693	-5.953885	-5.857076	-6.545031	-6.064242	-6.116370	-6.094219	-5.877597	-5.884045
FIMALG	d	0.3394	0.2893	0.4022	0.3841	0.4762	0.3917	0.3910	0.3807	0.0000	0.2122
	(se)	0.0442	0.0442	0.0319	0.0425	0.0395	0.0368	0.0344	0.0472	0.0023	0.0047
	LLik	20439.47	20738.75	18777.88	18446.95	20682.61	19270.02	19569.02	13585.37	18017.63	18147.56
	BIC	-6.493117	-6.588291	-5.964703	-5.859465	-6.545457	-6.065268	-6.118989	-6.094975	-5.876189	-5.885936

Table 1 - Log-Likelihood, BIC, long memory parameter estimate and its standard error for the Indices (part 1)

Model	Statistic	MEX	IPC	AEX	ATX	ISQ	KOR	MAD	NYS	PSI	SWI
G	LLik	21200.46	19564.98	19968.33	18958.00	19365.32	18626.08	19086.62	20511.05	20415.53	20511.11
	BIC	-6.747244	-6.231269	6.24243	-6.098047	-6.116919	-6.041762	-5.999405	-6.51727	-6.371389	-6.530792
FIG	d		1.0000	0.7709	0.5138	0.4885	0.8195	0.6363	0.6770	0.4737	0.6510
	(se)	f.t.c.	0.0417	0.0541	0.0425	0.0445	0.0689	0.0470	0.0528	0.0372	0.0512
	LLik		19474.98	19905.07	18962.39	19369.01	18517.53	19071.70	20497.93	20439.60	20506.11
	BIC		-6.201186	-6.221264	-6.098054	-6.116702	-6.005102	-5.993333	-6.511707	-6.377537	-6.527803
APA	LLik	21301.62	19635.20	f.t.c.	19051.08	19434.96	f.t.c.	19207.18	20634.17	20490.05	20674.24
	BIC	-6.776680	**-6.250865**		-6.125202	-6.136169		-6.034579	-6.553644	-6.391927	-6.579990
FIAPA	d	0.5000		0.5000	0.4290	0.4543	0.5000	0.5000	0.4575	0.4402	0.5000
	(se)	0.0352	f.t.c.	0.0355	0.0390	0.0399	0.0528	0.0461	0.0349	0.0400	0.0406
	LLik	21232.22		20035.45	19057.74	19441.13	18566.67	19166.35	20622.96	20500.47	20656.81
	BIC	-6.753181		-6.259318	-6.125942	-6.136736	-6.018224	-6.020354	-6.548688	-6.393813	-6.57304
EG	LLik	21289.54	19619.97	20117.81	19039.38	19426.94	18679.56	19194.13	20617.79	20483.62	20658.67
	BIC	-6.775619	-6.248798	-6.289200	-6.124248	-6.136399	-6.059125	-6.033227	-6.551216	-6.392656	-6.577813
FIEG	d	0.5000	0.5000	0.4415	0.4652	0.4573	0.5000	0.3903	0.4545	0.4008	0.0000
	(se)	0.0408	0.0447	0.0352	0.0428	0.0392	0.0388	0.0184	0.0339	0.0535	0.0004
	LLik	21300.96	19628.78	20123.13	19049.02	19450.26	18718.06	19204.28	20635.26	20490.73	20659.22
	BIC	**-6.777861**	-6.250215	**-6.289494**	**-6.125943**	-6.142388	-6.070210	**-6.035042**	-6.555380	-6.393508	-6.576552
MEG	LLik	21292.66	19616.68	20124.57	19038.01	19432.76	18683.79	19196.75	20626.14	20487.38	20659.08
	BIC	-6.776725	-6.247751	-6.291314	-6.123807	-6.138239	-6.060498	-6.034052	-6.553869	-6.393832	**-6.580698**
FIMEG	d	0.4915	0.5000	0.4178	0.4557	0.4386	0.5000	0.3709	0.4138	0.4224	0.0000
	(se)	0.0416	0.0408	0.0380	0.0450	0.0392	0.0570	0.0448	0.0359	0.0400	0.0517
	LLik	21300.95	19625.72	20122.82	19044.04	19452.42	18721.03	19203.88	20638.17	20493.19	20668.16
	BIC	-6.777860	-6.249237	-6.289397	-6.124340	**-6.143073**	**-6.071174**	-6.034918	**-6.556307**	**-6.394277**	-6.579445
MALG	LLik	21273.45	19599.77	20111.45	19027.78	19418.38	18666.20	19186.70	20601.47	20467.11	20652.89
	BIC	-6.776610	-6.242360	-6.287210	-6.120513	-6.133694	-6.054788	-6.030889	-6.546024	-6.387501	-6.575973
FIMALG	d	0.4278	0.4889	0.3637	0.3713	0.4187	0.5000	0.3368	0.3783	0.3190	0.0000
	(se)	0.0450	0.0447	0.0408	0.0539	0.0423	0.0812	0.0453	0.0379	0.0829	0.0014
	LLik	21277.73	19613.84	20106.23	19028.37	19436.65	18704.66	19191.22	20611.11	20469.37	20653.35
	BIC	-6.770462	-6.245450	-6.284205	-6.119295	-6.138088	-6.065859	-6.030934	-6.547701	-6.386838	-6.574727

Table 2 - Log-Likelihood, BIC, long memory parameter estimate and its standard error for the Indices (part 2)

It is important to note that, in most cases, all estimated coefficients are statistically significant at conventional levels (e.g., 1%), thereby supporting the inclusion of leverage and magnitude effects within the EGARCH framework, indicating that these components contribute meaningfully to capturing asymmetries and enhancing the overall accuracy of volatility estimation. Furthermore, across the examined exponential GARCH cases, the estimated autoregressive coefficient $\hat{\phi}$ is comparable in terms of their values and ranges from 0.6680 to 0.9668, indicating a high degree of persistence in volatility. The estimated coefficients associated with the magnitude effect $\hat{\gamma}$ are consistently positive, varying between 0.0990 and 0.2117, suggesting that the impact of shock magnitude on volatility is both statistically significant and economically relevant. As anticipated, the parameter $\hat{\kappa}$ capturing the asymmetric effect is consistently negative across all specifications, with relatively small absolute values ranging from 0.0730 to 0.2574. This outcome aligns with theoretical expectations, indicating that negative shocks have a slightly stronger impact on volatility than positive shocks of the same magnitude.

In unreported statistics, we can identify subtle differences in asymmetry and shock-response patterns between mature and emerging markets which are reflected in model nuances. Developed market indices often show a pronounced leverage effect. In emerging markets, the leverage effect can be present but is sometimes milder or at least different in nature – for example, if short-selling is restricted, negative shocks might not translate to volatility jumps as sharply. Additionally, some emerging markets experience occasional (political) crises or regime changes that could introduce structural breaks. While our models did not explicitly include regime-switching, the long memory component can partially proxy those slow mean shifts. It's possible that in such markets, the fractional long memory parameter estimate $\hat{d}$ captures both genuine long-range dependence and the effect of unresolved breaks. An indication of this is the fact that 2 of the 3 long memory parameters that fall out of the stationary range belong to the emerging market sector.

Overall, the empirical findings suggest that the recently developed models within the EGARCH family offer a compelling alternative to conventional parametric volatility models for the purpose of market risk measurement. These models are particularly valuable in cases where standard GARCH-type specifications exhibit poor performance or fail to capture key features of the data. In such instances, it is advisable to consider power- and modulus-transformed exponential GARCH models as they can yield more accurate and stable volatility estimates. Their flexibility in modeling volatility makes them especially well-suited for quantitative risk management (Value at Risk and Expected Shortfall).

Index - Model	$\widehat{\omega}$	$\hat{\phi}$	$\hat{\kappa}$	$\hat{\gamma}$	$\hat{d}$	n
S&P - FIMEGARCH	-8.7479	0.7955	-0.2412	0.1520	0.3807	6289
	(0.1138)	(0.0324)	(0.0147)	(0.0106)	(0.0357)	
NSQ - FIEGARCH	-8.2978	0.7271	-0.1166	0.1230	0.4763	6289
	(0.1331)	(0.0474)	(0.0116)	(0.0118)	(0.0353)	
RUS-FIMEGARCH	-8.4540	0.8218	-0.1541	0.1083	0.4161	6289
	(0.1019)	(0.0337)	(0.0140)	(0.0093)	(0.0462)	
FTS-FIEGARCH	-9.1924	0.6917	-0.1304	0.1239	0.5000	6313
	(0.2866)	(0.1220)	(0.0252)	(0.0142)	(0.0629)	
CAC - FIEGARCH	-8.7337	0.7675	-0.1284	0.0990	0.4578	6389
	(0.0912)	(0.0337)	(0.0090)	(0.0097)	(0.0298)	
EST - FIEGARCH	-8.7528	0.7548	-0.1475	0.1027	0.4508	4451
	(0.1042)	(0.0436)	(0.0117)	(0.0112)	(0.0385)	
NIK - MEGARCH	-0.3511	0.9590	-0.1827	0.2038		6125
	(0.0463)	(0.0054)	(0.0240)	(0.0132)		
HSI - FIMEGARCH	-7.9696	0.8169	-0.0730	0.1163	0.4565	6159
	(0.1876)	(0.0457)	(0.0111)	(0.0108)	(0.0576)	
MEX – FIEGARCH	-9.0729	0.7660	-0.0884	0.1248	0.5000	6279
	(0.1595)	(0.0421)	(0.0083)	(0.0107)	(0.0408)	
AEX - FIEGARCH	-8.8994	0.7750	-0.1236	0.1183	0.4415	6392
	(0.0993)	(0.0372)	(0.0089)	(0.0104)	(0.0352)	
ATX - FIEGARCH	-8.5576	0.6995	-0.0956	0.1665	0.4652	6212
	(0.1311)	(0.0580)	(0.0091)	(0.0153)	(0.0428)	
ISQ - FIMEGARCH	-8.4263	0.7534	-0.1463	0.1423	0.4386	6326
	(0.1137)	(0.0436)	(0.0143)	(0.0124)	(0.0392)	
KOR - FIMEGARCH	-8.0963	0.6219	-0.1774	0.1779	0.5000	6160
	(0.1810)	(0.1266)	(0.0309)	(0.0232)	(0.0570)	
MAD - FIEGARCH	-8.5947	0.8269	-0.0917	0.1205	0.39028691	6357
	(0.1037)	(0.0200)	(0.0080)	(0.0111)	(0.01842625)	
NYS - FIMEGARCH	-8.8315	0.7979	-0.2040	0.1391	0.4138	6289
	(0.1223)	(0.0317)	(0.0137)	(0.0101)	(0.0359)	
PSI - FIMEGARCH	-8.6584	0.6680	-0.1611	0.2117	0.4224	6403
	(0.1044)	(0.0686)	(0.0181)	(0.0222)	(0.0400)	
SWI – MEGARCH	-0.3065	0.9668	-0.2574	0.1593		6276
	(0.0298)	(0.0032)	(0.0142)	(0.0119)		

Table 3 - Model Coefficients of the Optimal exponential GARCH Models selected by BIC

Index - Model	$\widehat{\omega}$	$\hat{\phi}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	n
DJI - APARCH	0.0002	0.0905	0.9004	0.8668	1.0650	6289
	(0.0000)	(0.0073)	(0.0085)	(0.0819)	(0.0490)	
DAX- APARCH	0.0003	0.0694	0.9216	0.9999	1.0349	6347
	(0.0000)	(0.0047)	(0.0056)	(0.0252)	(0.0751)	
IPC - APARCH	0.0000	0.0693	0.9234	0.5367	1.3709	6274
	(0.0000)	(0.0069)	(0.0066)	(0.0651)	(0.130)	

Table 4 - Model Coefficients of the Optimal APARCH Models selected by BIC

In the case of the DAX the estimated value of the asymmetry coefficient $\hat{\gamma}$ of the selected APARCH model is close to its theoretical upper bound of 1. This boundary implies a scenario in which positive returns from the previous trading day exert no influence on the current day's volatility - an effect referred to as a perfect leverage effect, meaning that volatility responds almost exclusively to negative shocks. In contrast, for the IPC the estimated asymmetry coefficient is $\hat{\gamma} = 0.5367$. According to this APARCH specification, the relative impact of a negative return compared to a positive one on the volatility can be quantified by $\left(\frac{1+\hat{\gamma}}{1-\hat{\gamma}}\right)^{\hat{\delta}}$. This observed phenomenon is particularly relevant for risk management and volatility forecasting, as it highlights the importance of incorporating asymmetry for accurate representation of market dynamics.

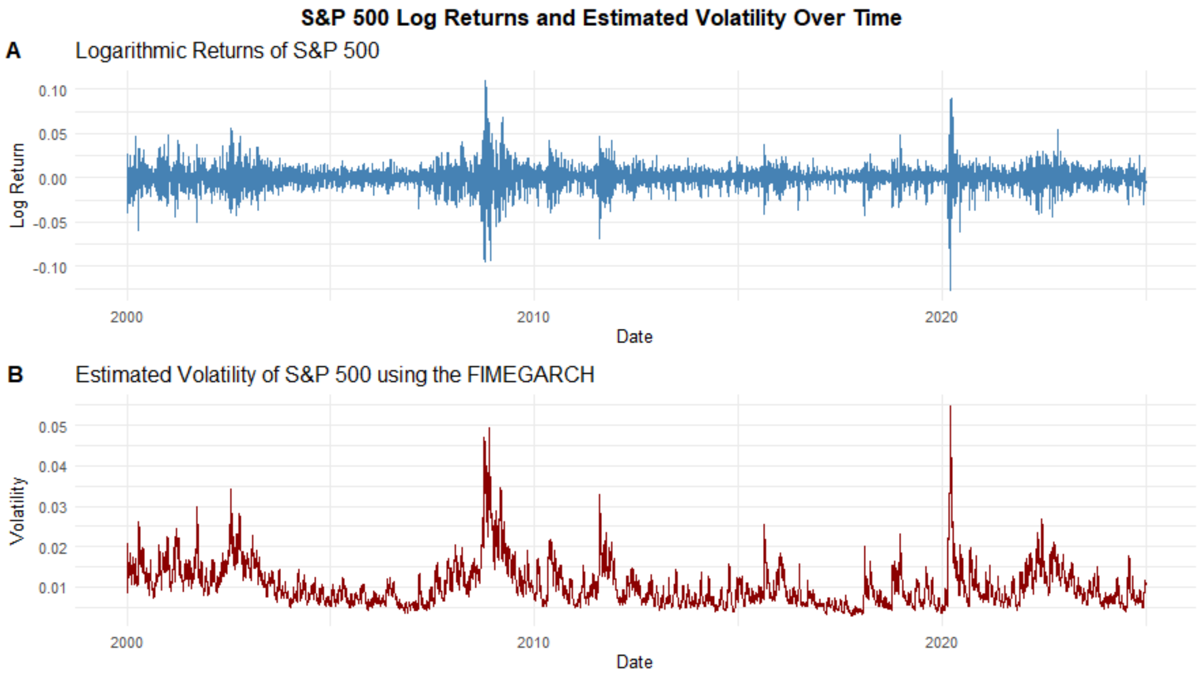

Figure 3 - S&P 500 Log Returns and Estimated Volatility from January 2000 to December 2024

Figure 3 presents the logarithmic returns and the corresponding estimated volatility using the selected FIMEGARCH model over the sample period. The estimated volatility series closely mirrors the turbulent periods with pronounced peaks observed during the dot-com bubble burst, the global financial crisis, and the COVID-19 shock. Between these events, volatility tends to remain relatively subdued, though elevated volatility is observed during periods of heightened macroeconomic or geopolitical uncertainty. It is noteworthy that periods characterized by exceptionally high absolute returns tend to coincide with contemporaneous spikes in the volatility estimate (depicted in Figure 4), a well-documented empirical phenomenon whereby financial market turbulence - manifested through large positive or negative price movements - is typically accompanied by heightened investor uncertainty and risk aversion, leading to increased volatility. Such co-movement highlights the role of volatility estimates as forward-looking measures of market sentiment and perceived risk.

While only the volatility estimates of the FIMEGARCH for the S&P 500 return series are demonstrated, the volatility curves estimated by the other volatility models exhibit only slight trajectory differences, which can be attributed to the distinct structural characteristics and assumptions underlying each volatility model. For instance, while all basic GARCH-type models emphasize autoregressive dynamics in volatility by incorporating lagged returns, processes of the EGARCH family depend on the unexpected innovations. Despite these methodological differences, the resulting volatility estimates tend to follow a broadly similar pattern, particularly during periods of significant market stress.

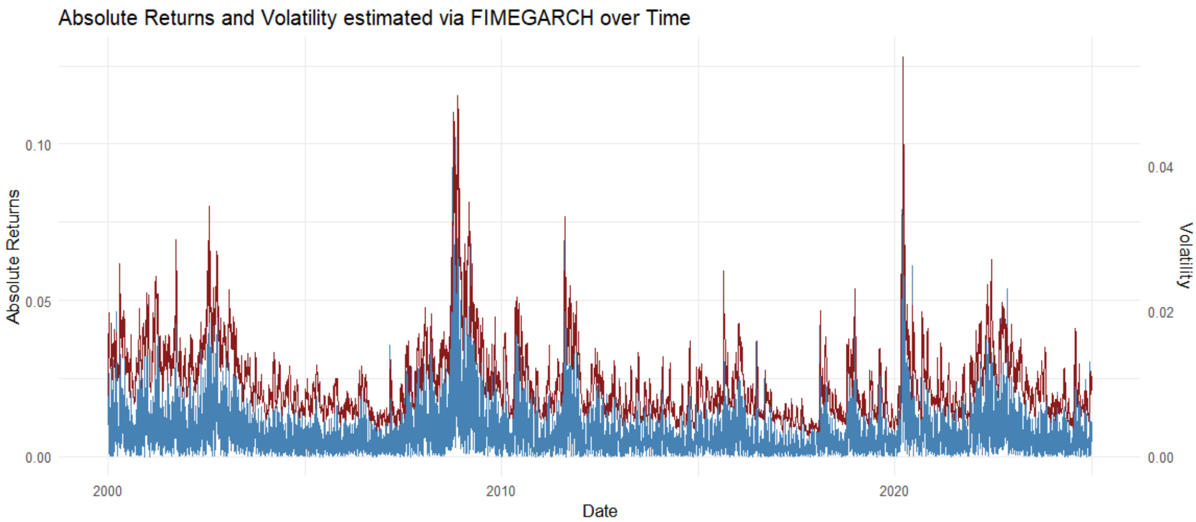

Figure 4 - S&P 500 Absolute Returns with corresponding, estimated volatility process

There are fundamentally two different approaches to volatility estimation. Historical volatility - derived from past price data - is typically estimated using econometric models, such as the GARCH-type models. In contrast, implied volatility is primarily employed in the context of option pricing theory. It reflects the market's current expectations of the future variability of an underlying asset over the remaining life of the corresponding option. Implied volatility is inferred from observed option prices and, together with other relevant parameters, can be derived using the Black-Scholes model (Black and Scholes, 1973) or similar pricing frameworks via numerical approximation techniques. Therefore, while historical volatility is backward-looking and data-driven, implied volatility is forward-looking and encapsulates market sentiment and expectations.

Figures 4 and 5 present a comparative analysis of these two different approaches to estimating market volatility. Figure 4 compares the S&P 500 absolute returns with volatility estimates generated by the parametric optimal model, while Figure 5 contrasts the same absolute returns with the implied volatility index (CBOE VIX). The visual comparison clearly reveals that the volatility estimated via the FIMEGARCH model aligns more closely with the realized market behavior, as represented by the absolute returns than the VIX does. This can be particularly observed during periods of financial turbulence. For instance, in the aftermath of the dot-com bubble, the FIMEGARCH model captures the prolonged elevation in volatility with greater precision, whereas the VIX appears to lag in its response. Similarly, during the initial phases of the Global Financial Crisis and the onset of the COVID-19 pandemic, the GARCH-based model exhibits a more immediate and sharper reaction to market stress, whereas the implied volatility index reflects a more delayed and smoothed trajectory.

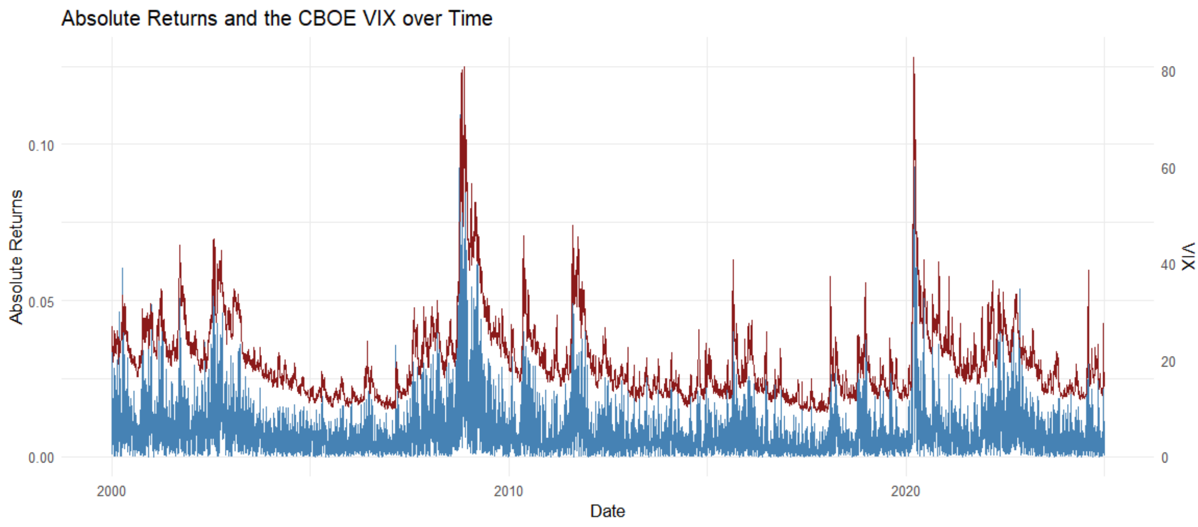

Figure 5 - S&P 500 Absolute Returns with corresponding CBOE VIX volatility series

This discrepancy is attributed to fundamental differences in model design. The FIMEGARCH model is backward-looking and relies on historical return data, allowing it to adapt dynamically to recent market shocks through its volatility updating mechanism. In contrast, the VIX is derived from option prices and reflects market participants' expectations of future volatility over a fixed horizon (typically 30 days). As such, the VIX reacts more conservatively or with a delay, especially during sudden spikes in uncertainty. Therefore, the comparison underscores the strengths of parametric volatility models in closely tracking realized market fluctuations. At the same time, it highlights the complementary nature of implied volatility indices, which offer forward-looking insights but may exhibit inertia in the face of abrupt market changes.

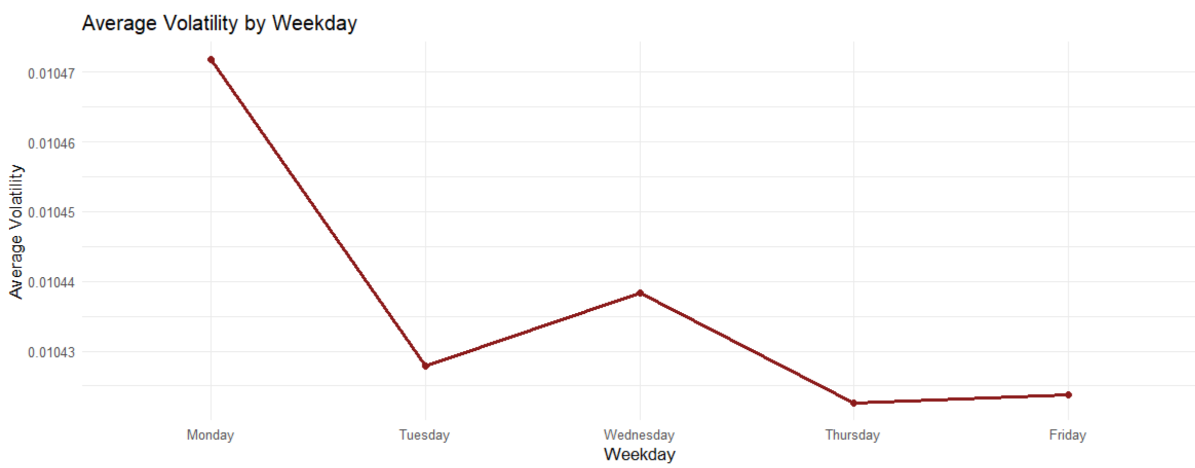

Figure 6 - S&P 500's average Volatility by Weekday from January 2000 until December 2024

Figure 6 illustrates the average daily volatility of the S&P 500 by weekday, calculated using the FIMEGARCH model estimates over the 25-year sample period, and reveals a systematic pattern in volatility dynamics throughout the week. Both the model-based volatility estimates and the implied volatility measure (VIX, not shown here for reasons of saving space) exhibit a consistent trend: volatility tends to decline as the week progresses, consistent with empirical findings in the literature (e.g., Fleming et al., 1995). Several explanations have been proposed for this behavior, including the accumulation of information over the weekend, which may contribute to elevated uncertainty at the beginning of the week, and the gradual resolution of uncertainty as more market-relevant news becomes available during the week. Additionally, reduced trading activity and lower news intensity toward the end of the week may contribute to the decline in volatility.

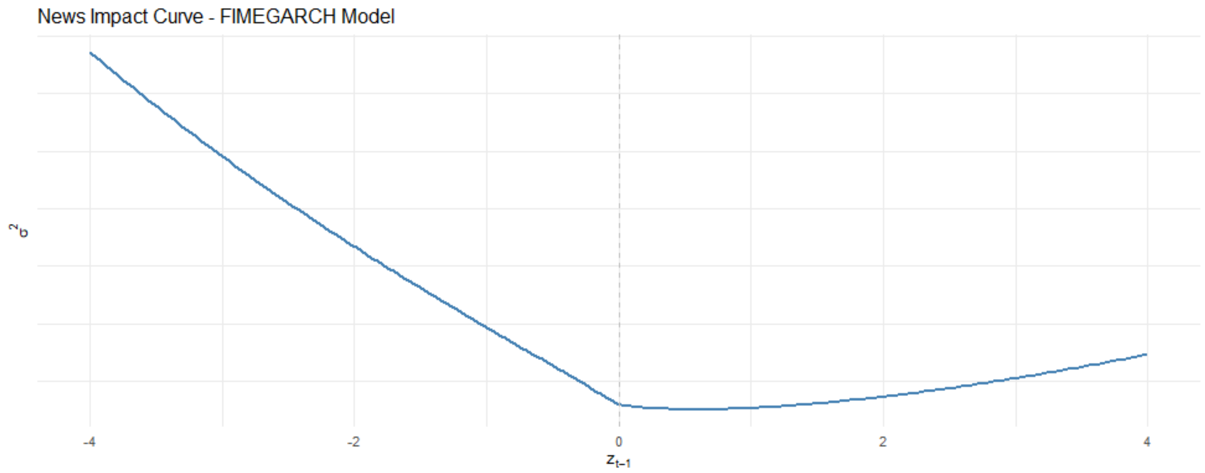

Figure 7 - News Impact Curve for the S&P 500 of the FIMEGARCH

Figure 7 illustrates the news impact curve derived from the FIMEGARCH model which depicts the effect of past standardized shocks on current conditional variance. The curve exhibits clear asymmetry, indicating the inherent leverage effect. This pattern reflects the modified exponential structure of the selected model, which allows for greater sensitivity to large negative returns.

It is worth noting and well-documented in the literature (e.g. Sibbertsen, 2004) that structural breaks, deterministic trends, or regime shifts can sometimes mimic long memory, creating a spurious artifact. Except for the cases of FTS, MEX, and KOR, the estimated long memory parameter $\hat{d}$ falls within the stationary range across the time series. A detailed analysis of these price processes suggests that the observed non-stationarity may be attributed to pronounced structural breaks in the unconditional variance during the three respective crises depicted above. While the shifts are not indicative of an underlying stochastic non-stationarity, they rather reflect a deterministic regime change in market conditions. Consequently, the elevated persistence estimate may be spuriously inflated due to this abrupt break, highlighting the importance of accounting for structural changes when interpreting long memory behavior in volatility. Panel A and B of Figure 8 illustrate these shifts for the example KOR, showing a clear distinction between the unconditional variance before and after the crisis periods. Therefore, if the structural break in the KOR series is properly detected, estimated, and accounted for, the corresponding $\hat{d}$ parameter would likely fall within the stationary range.

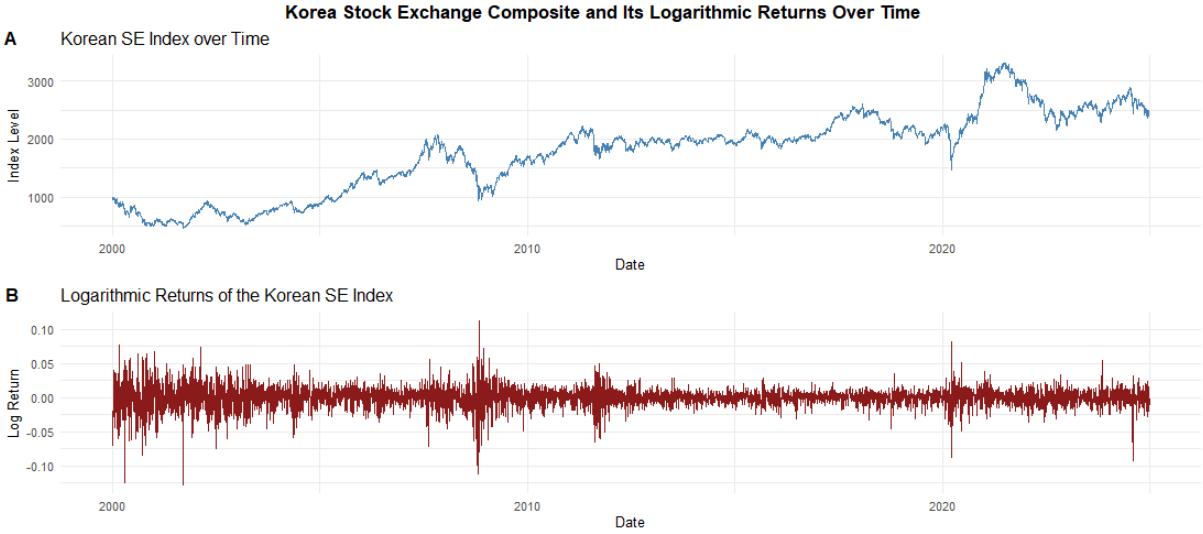

Figure 8 – Daily Korea S.E. Composite Index with Log-Returns from January 2000 until December 2024

Consequently, it is important to notice that forecasting without incorporating such structural breaks may result in misleading conclusions. The problem of slowly changing scale functions in the conditional volatilities can be solved by using a semiparametric extension of the proposed (long-memory) GARCH models, a procedure which was initiated by Letmathe et al. (2022). As noted by Letmathe et al. (2022), the estimated long memory parameter tends to assume smaller values in semiparametric models. This outcome is primarily attributable to the fact that such models first isolate and remove the deterministic component from the observed time series before applying a parametric model to the remaining stochastic structure. Consequently, if a deterministic trend or scale function is not properly accounted for, it may be mistakenly interpreted as long-range dependence or short-term persistence in the data. This misattribution can result in biased parameter estimates and unstable volatility forecasts, ultimately impairing the reliability of derived risk measures such as Value at Risk and Expected Shortfall. Further investigation of this explanation will be part of future research.

Figure 9 displays the standardized residuals obtained after fitting the FIMEGARCH model to the S&P 500 return series. As a crucial component of model adequacy assessment, this diagnostic plot indicates that the model has effectively captured the conditional heteroskedasticity inherent in financial return data through visual inspection. The residuals are centered around zero and exhibit a relatively constant variance over time, suggesting that the model has adequately accounted for time-varying volatility. The absence of systematic volatility clustering or visible patterns supports the assumption of a white noise process in the standardized residuals While a few extreme observations remain, they appear isolated and infrequent; they do not persistently cluster, which reinforces the conclusion that the FIMEGARCH model has successfully filtered out the conditional variance dynamics. Further inspection of the raw and squared standardized residuals underlines this conjecture as they do not show any significant autocorrelation beyond lag 0, which indicates that both the mean and the volatility dynamics have been adequately accounted for, the conditional heteroskedasticity has been sufficiently filtered out and the residuals form an uncorrelated white-noise process. Therefore, there are no remaining ARCH effects present in the data, which validates the adequacy of the FIMEGARCH specification. Similar results can be obtained for all the other specified and fitted models, but are not reported to save space.

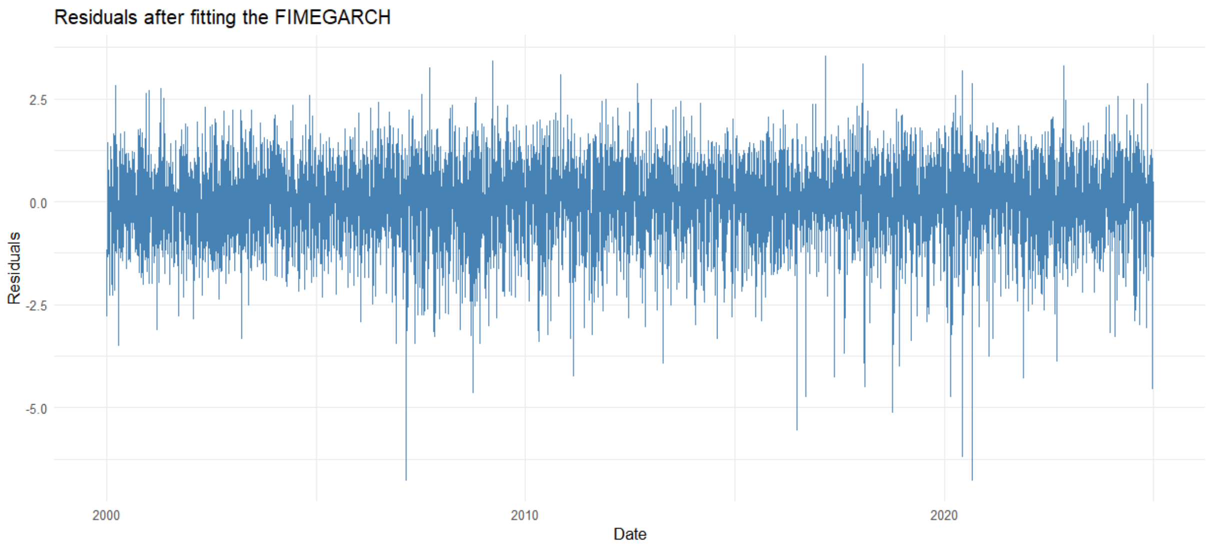

Figure 9 – Standardized Residuals of the S&P 500 Return Series after fitting the FIMEGARCH

Overall, the empirical evidence presented in this study demonstrates that models within the general EGARCH family exhibit strong performance in capturing conditional heteroscedasticity in financial return series, offering a valuable alternative to well-established volatility processes, combining strong theoretical properties with solid practical performance. However, the results also underscore the importance of market- and series-specific characteristics - such as the strength of asymmetries (leverage effects) or the presence of extreme observations - which can influence the relative performance of different model specifications. The heterogeneity suggests that no single GARCH-type model universally outperforms across all markets. While FIMEGARCH and FIEGARCH models frequently provided the best fit, some indices were more accurately described by alternative formulations, such as the power-transformed APARCH. These findings reinforce the need for both researchers and practitioners to systematically compare multiple GARCH-family variants rather than relying solely on "vanilla" GARCH models. A flexible, data-driven model selection strategy is essential for accurately capturing the volatility dynamics specific to each market context.

6 Conclusion

The paper at hand introduces the general framework of the exponential GARCH family and investigates specific model cases under the conditional normal distribution. A broad comparison is conducted using short and long memory versions of prominent models within the EGARCH family. These are benchmarked against traditional models - including (FI-)GARCH and (FI-)APARCH - using a dataset of twenty major international stock indices. The comparative empirical analysis demonstrates that the general EGARCH framework constitutes a compelling alternative to conventional short and long memory volatility models, as EGARCH-type models frequently outperform their non-exponential counterparts in both statistical applicability and practical performance, primarily due to their improved statistical properties, such as robustness to asymmetry and better handling of persistence in volatility. These findings underscore the advantages of (long memory) exponential specifications in capturing key stylized facts of financial return series, including volatility asymmetry and long memory behavior, while addressing several well-known limitations of standard GARCH-type models.

Overall, the findings of this study corroborate the widely accepted view in the economic and financial time series literature that long memory models tend to outperform their short memory counterparts as, in many markets adjustments to shocks occur gradually rather than instantaneously. As a result, the initial impulse can influence market dynamics over extended periods, leading to persistent autocorrelations in the data. In financial markets, full and immediate adjustment to new information is impeded by frictions such as information asymmetries, transaction costs, institutional rigidities, and the time-consuming nature of restructuring processes, which can lead not only to long memory, but also to dual long memory or to deterministic smooth trends embedded in the data. Recognizing and modeling these features is essential for capturing the true underlying market dynamics. From a policy-making perspective, the implications are substantial as the implementation of more accurate volatility models leads to more reliable inference and contributes to more robust and effective decision-making in both public policy and private sector finance. In financial markets, the presence of long memory, dual long memory, or deterministic trends can be exploited to improve the forecasting of asset prices, returns, or risk measures. Accounting for these features can reduce market inefficiencies, improve risk management strategies, and support more informed investment decisions. Therefore, the integration of long memory characteristics into economic and financial modeling should be regarded as a vital component of modern empirical analysis and forecasting. While this paper only focuses on univariate parametric long memory in volatility, future research should investigate semi-parametric extensions. Empirical evidence suggests that volatility series exhibiting long memory behavior may also be subject to structural breaks or slowly evolving deterministic components, such as time-varying volatility levels or regime shifts. Accurate and efficient estimation of the long memory parameter in such cases is only possible if these deterministic features are appropriately accounted for. Failure to control for structural breaks or smooth trends may result in biased parameter estimates and misidentification of the true degree of persistence, thereby undermining the reliability of volatility modeling and forecasting. The evidence at hand supports the adoption of flexible, exponential long memory volatility models as valuable tools for modeling financial time series and improving forecast accuracy. Even so, model performance is inherently market-dependent, as the effectiveness of a given specification may vary across different financial environments and asset classes. As such, it is recommended that both risk managers and regulatory authorities regularly monitor, validate, and benchmark a diverse set of volatility models to ensure that risk assessments remain accurate and robust under varying market conditions, thereby enhancing the reliability of forecasts and the effectiveness of risk management frameworks.

Nonetheless, some methodological challenges remain. For instance, issues such as the invertibility of the exponential volatility processes and the theoretical properties of the general EGARCH framework warrant further research. Addressing these open questions could further strengthen the practical applicability of exponential long memory models in financial econometrics.

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